Quantile hedging pension payoffs: an analysis of investment incentives[*]

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October 18, 2017
(Accepted version)

Abstract

Pension payoffs that depend on the term structure of interest rates can be superhedged by trading only in the bond market, albeit at a possibly high initial price. Quantile hedging techniques can be used to reduce the cost of the strategy if the investor is willing to accept the risk that, with low probability, the hedging portfolio will not be sufficient to cover the payoff. In this paper, we use results on partial hedging of defaultable claims to assess whether the cost of a quantile hedging strategy can be reduced by the addition of a risky asset to the bond market. In a simplified market model, we derive explicit expressions for the initial cost of the hedge and the expectation of the unhedged loss. We show that while investing in the risky asset reduces the cost of the hedge, it can significantly increase the risk linked to the unhedged loss. In the context of pension plan funding, we thus provide an example of the risky investment strategies encouraged by a partial hedge criterion based on the probability of loss.

1 Introduction

We consider a bond market, in which we seek to hedge a payoff linked to the term structure of interest rates and contingent on the survival of an individual. Setting up a superhedging portfolio would ensure that there is always sufficient funds to pay the claim at maturity. The initial cost of such a hedging strategy is often very high.

An investor may not always be willing to spend the full superhedging price on a hedging strategy. In exchange for investing a smaller amount, she may accept to take some risks. Quantile hedging, introduced by Föllmer and Leukert (1999), describes hedging strategies that cover a payoff with maximal probability, at a fixed cost which is lower than the superhedging price. In this paper, we consider a pension payoff for which a quantile hedge can be constructed using only bonds, for a cost lower than the cost of a superhedge. However, the question we seek to answer here is whether adding a risky asset to the hedging portfolio can further reduce the initial cost of

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the quantile hedging strategy. We also want to highlight the possible risks involved with such a hedging criterion.

When considering only superhedging strategies for the pension payoff, it will be clear that the additional risky asset is useless to hedge a claim that only depends on the term structure of interest rates. However, if the investor is willing to consider a hedge that only covers the payoff with high probability (but still less than 1), is it possible to further reduce the price of the replicating portfolio by also investing in equity?

This question is relevant to pension plan funding. Apart from demographic risks, the payoffs linked to defined benefit plans or hybrid plan designs, such as cash balance (Hardy et al. Hardy, Saunders, and Zhu (2014)), depend mostly on the evolution of the term structure of interest rates. In particular, some cash balance plans credit guaranteed returns expressed as functions of a spot rate.

In the industry, these payoffs are however often hedged using a portfolio which is significantly invested in equity. This can be explained by a few different factors. First, while investments in equity are generally riskier, they yield higher returns on average. This additional return, compensating for the additional risk taken on by investing in equity, is called *equity risk premium*. There is some evidence of its existence (Dimson, Marsh, and Staunton (2003)), but there is no agreement on its size or its behaviour through time. Nonetheless, on longer horizons, it is generally accepted that equity will yield a higher average return than bonds, which may motivate the composition of pension funds. Second, since stock and bond prices tend to be negatively correlated, particularly in bear markets (see, for example, Yang, Zhou, and Wang (2009)), investing in equity may provide a hedge for interest rate linked payoffs. Therefore, it is relevant to ask whether the pension industry's investments in equity are supported by the theory of quantile hedging.

It is also crucial to assess the risk resulting from a quantile-type hedging criterion. Indeed, a criterion that incentivizes investment strategies that have a small probability of resulting in a big loss might not be appropriate for the pension and life insurance industry, since it is responsible for the financial security of a large number of retirees. Nonetheless, by focusing on the probability of a loss, regulation based on risk measures such as the Value-at-Risk (VaR) may create the same type of incentive. Therefore, the study of the investment incentives linked to quantile hedging in the pension industry can also be a further example of potential problems with regulation that only take the probability of loss into account.

General results on quantile hedging were first presented in Föllmer and Leukert (1999). Sekine (2000) and Nakano (2011) obtained results pertaining to the partial hedging of defaultable claims. As those results also apply in the context of survival contingent payoffs, we review them in this paper. Quantile hedging has been used to hedge different derivatives in various markets. For example, Krutchenko and Melnikov (2001) develop explicit formulas to hedge options in a jump-diffusion market. Many authors also use quantile hedging in the context of life insurance. In particular, Melnikov and Skornyakova (2005) apply quantile hedging to pure endowment equity-linked contracts, Gao, He, and Zhang (2011) generalize these results to a bigger class of equity-linked insurance contracts, and Wang (2009) focuses on death benefits.

Melnikov and Tong (2012, 2013, 2014) study similar problems in markets with stochastic interest rates and transaction costs. Also in life insurance, Klusik and Palmowski (2011) explore the incompleteness brought on by insurance risks such as mortality.

Quantile hedging was extended to more general partial hedging strategies in Föllmer and Leukert (2000). In particular, these strategies take into account the size of the loss. In that mindset, Barski (2016) provides a refinement of the strategy which also limits the loss, while working with a slightly different definition of admissible strategies. Finally, Cong, Tan, and Weng (2013, 2014) also present hedging approaches based on the distribution of the unhedged loss. In this paper, we focus mainly on the results of Föllmer and Leukert (1999) and Sekine (2000), and reserve the application of further partial hedging approaches for future work.

Most of the literature applying quantile hedging to life insurance is concerned with the risk management of equity-linked payoffs by trading in a (more or less) general equity market. Our approach differs in this way: the payoff we seek to hedge is only linked to interest rates, and could be superhedged by only investing in bonds. However, we extend the market to include a risky asset, and assess its impact on the cost of the strategy. Mathematically, we seek to hedge the claim using a filtration that is richer than the sub-filtration on which the claim is defined. To the author's knowledge, this application of quantile hedging has not previously been explored.

An important motivation for our work is Hardy, Saunders, and Zhu (2014), which demonstrates that the (risk-neutral) market value of a cash balance plan benefit is typically much higher than the value actuaries assign to the payoff for plan funding purposes. In this paper, we attempt to explain this seemingly underfunded position by two factors. First, we assume that the plan sponsor is willing to take a small chance of a loss when the claim comes to maturity. Second, we permit investment in a risky asset, which allows the hedger to make use of the equity risk premium to reduce the cost of the hedge. These assumptions seem in line with the funding practices of pension plan sponsors.

Our results show that, in the simplified market we use to derive explicit expressions, the availability of a risky asset is in fact effective in reducing the cost of a quantile hedge. However, we also demonstrate that the optimal quantile hedge invested in part in equity can be significantly riskier than its bond-only counterpart. Therefore, a hedging criterion defined in terms of the probability of loss may give an incentive to implement riskier hedging strategies by reducing their prices, which is not desirable in the pension industry, where the failure of a hedging strategy can have a significant impact on the financial security of large groups of retirees.

To study the problem described above, we define two markets: in the first one, bonds are the only assets available for trading. The second market is constructed by adding a risky asset to the first one. Using quantile hedging results for defaultable claims presented in Sekine (2000), we derive the cost of the optimal hedging strategy in each market.

In Section 2, we introduce the two markets in which we compare the cost of the quantile hedge, and we present the payoff we seek to hedge. In Section 3, quantile hedging results are reviewed and applied to our setting. This allows us to compare the costs and the riskiness of the quantile hedge in both markets. Section 4 contains numerical illustrations and Section 5 concludes.

2 Setting

Our main goal in this paper is to compare the cost of a partial hedging strategy for an interest rate linked payoff in two different market settings. In the first market, bonds are the only assets available for trading, while in the second one, investors can also trade in a risky asset. In this section, we present these two markets in further details, discuss mortality risk and introduce the payoff we consider throughout the rest of the paper.

2.1 Financial markets

2.1.1 Bond-only market

We work on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathbb{P}$ denotes the real-world (objective) measure. We also fix a time horizon $T \in \mathbb{R}^+$, which will represent the maturity of a payoff, and a final time horizon $2T$. We extend the time horizon beyond the maturity of the payoff to ensure that the prices of bonds with time to maturity of at most T are well defined up to time T.
We consider a bond market in which at each time t, $0 \leq t \leq T$, bonds of all times to maturity $\bar{\tau} \in [0, 2T - t]$ are traded. Then, from times 0 to T, it is always possible to trade bonds with time to maturity T (thus coming to maturity at the latest at time $2T$). We denote the price at time t of a bond maturing at s by $P(t, s)$, for $0 \leq t \leq T$, $0 \leq s \leq 2T$ with $t \leq s$. To exclude arbitrage, we assume that these prices are strictly greater than 0 with $P(t, t) = 1$, for $0 \leq t \leq T$. The short rate process $\{r(t)\}_{0 \leq t \leq T}$ can be extracted from the bond market. To do so, we define the instantaneous forward rate at time t for maturity s, denoted by $f(t, s)$, by

$$f(t, s) \coloneqq -\frac{\partial \log P(t, s)}{\partial s},$$

under the assumption that bond prices are sufficiently smooth. The instantaneous forward rate is the rate prevailing at time t for the infinitesimal time period $[s, s + ds)$, $t \leq s$. The short rate $r(t)$ is then defined by

$$r(t) \coloneqq f(t, t).$$

We also assume the existence of a bank account accumulating at the short rate. The value process of this asset is denoted by $\{B(t)\}_{0 \leq t \leq T}$, so that

$$B(t) = e^{\int_0^t r(s) ds}.$$

We let the short rate be modeled by an Ornstein-Uhlenbeck process:

$$dr(t) = a(b - r(t))dt + \sigma_r \, dW_r(t), \tag{2.1}$$

for $t \geq 0$, where a, b and σ_r are positive constants, and $W_r = \{W_r(t)\}_{t \geq 0}$ denotes a P-Brownian motion. This model is known in the literature as the Vasiček model (Vasiček, 1977). We further define $\mathbb{F}^B = \{\mathcal{F}_t^B\}_{t \geq 0}$ as the filtration generated by W_r and augmented by the P-null sets of $\mathcal{G}$. We also assume that the market price of interest risk, denoted by θ_r, is unique and constant, so that the price $P(t, s)$ at time t of a bond with maturity s is given by

$$P(t, s) = \exp\{\gamma(s - t) - r(t)D(s - t)\}, \tag{2.2}$$

where

$$D(s-t) = \frac{1-e^{-a(s-t)}}{a},$$
$$\gamma(s-t) = \left(b + \frac{\sigma_r\theta_r}{a} - \frac{\sigma_r^2}{2a^2}\right)(D(s-t)-(s-t)) - \frac{\sigma_r^2}{4a}(D(s-t))^2. \tag{2.3}$$

This result can be found in numerous references, one of which is Brigo and Mercurio (2007).
It follows that under the P-measure, the bond price has the following dynamics:

$$\frac{dP(t,s)}{P(t,s)} = r(t)dt - \sigma_{(s-t)}(dW_r(t) - \theta_r\ dt), \tag{2.4}$$

where $\sigma_{(s-t)} = \sigma_r D(s-t)$. In particular, the negative sign in front of $\sigma_{(s-t)}$ highlights the fact that bond prices rise when interest rates fall, and vice and versa.
For $0 \le t \le T$, we define the process $Z^B = \{Z_t^B\}_{0\le t\le T}$ by

$$Z_t^B = e^{\theta_r W_r(t) - \frac{1}{2}\theta_r^2 t}, \tag{2.5}$$

and observe that it is a strictly positive local martingale with $\mathbb{E}\left[Z_t^B\right] = 1$ for any $0 \le t \le T$. Therefore, we can define a new probability measure equivalent to P on $(\Omega, \mathbb{F}^B)$ by $\frac{dP^B}{dP}\big|_{\mathcal{F}_T} = Z_T^B$. It can be shown (for example, using Girsanov's Theorem and (2.4)) that under P^B, discounted bond prices are local martingales. It follows that P^B is an equivalent local martingale measure (EMM).
By Theorem 4.9 of Filipovic (2009), since the market price of risk θ_r is unique, the EMM defined by (2.5) is unique and our bond market is complete with respect to the filtration $\mathbb{F}^B$. It follows in particular that any $\mathcal{F}_T^B$-measurable claim with finite expectation can be perfectly replicated by trading in the bond market. The initial cost of this strategy is given by the P^B-expectation of the discounted payoff, its unique *no-arbitrage price.*

Remark 1. Since we are working with a one-factor short-rate model, the assumption that bonds of all maturities are available for trading at all times is not necessary. In fact, the bank account and a bond with maturity $2T$ is sufficient to replicate all the other bonds in our market. Therefore, going forward we can assume that those are the only two assets available for trading in the bond market, without losing its completeness.

2.1.2 Mixed market

The main goal of this paper is to assess whether the availability of equity can reduce the cost of imperfect hedging strategies for a pension-type claim, which is linked to the bond market and paid if the plan member reaches a certain age. The financial part of the payoff will be represented by an $\mathcal{F}_T^B$-measurable random variable with finite expectation, and it will therefore be possible to superhedge it by trading only in the bond market. Nonetheless, to account for the possibility of investing in equity, we introduce a second market, called the *mixed market*, which we obtain by adding a risky asset to the bond market, and by enlarging the filtration appropriately. In Section 3, we will compare the cost of hedging strategies developed in each of the two markets (bond-only and mixed).

The mixed market is also defined on the space $(\Omega, \mathcal{G}, P)$, and it is obtained by enlarging the filtration $\mathbb{F}^B$ as follows. Let $S = \{S_t\}_{0 \leq t \leq T}$ denote the price process of the risky asset discounted by the bank account numéraire, with P-dynamics given by

$$\frac{dS(t)}{S(t)} = \sigma_1 \left(dW(t) + \theta \; dt \right) - \sigma_2 \left(dW_r(t) - \theta_r \; dt \right), \tag{2.6}$$

where θ, σ_1 and σ_2 are constants with $\sigma_1 \neq 0$. θ_r is the market price of interest rate risk defined in Section 2.1.1. The process $W = \{W(t)\}_{t \geq 0}$ denotes a Brownian motion independent of W_r. If $\sigma_2 \neq 0$, then the risky asset discounted by the bank account numéraire is correlated to the interest rate through the process W_r. Note that σ_2 can be either positive or negative. When σ_2 is positive, then S is positively correlated with bond prices (this can easily be seen by comparing (2.4) and (2.6)).

We let $\mathbb{F}^S = \{\mathcal{F}_t^S\}_{0 \leq t \leq T}$ denote the filtration induced by W and augmented by the P-null sets of G. Finally, we can define $\mathbb{F}^M$ as the filtration generated by the mixed market, with $\mathcal{F}_t^M = \mathcal{F}_t^B \vee \mathcal{F}_t^S$, $0 \leq t \leq T$.

For $0 \leq t \leq T$, we define the process $Z^M = \{Z_t^M\}_{0 \leq t \leq T}$ by

$$Z_t^M = e^{\theta_r W_r(t) - \theta W(t) - \frac{1}{2}(\theta_r^2 + \theta^2)t}. \tag{2.7}$$

Since the process defined in (2.7) is a strictly positive local martingale with $\mathbb{E}\left[Z_t^M\right] = 1$ for $0 \leq t \leq T$, it can be used to define a new probability measure P^M equivalent to P on $(\Omega, \mathbb{F}^M)$ by $\frac{dP^M}{dP}\big|_{\mathcal{F}_T} = Z_T^M$. Simple calculations show that under this new measure, the discounted bond and risky asset price processes are local martingales. Therefore, P^M is an EMM.

For the filtration $\mathbb{F}^M$, which is induced by the two-dimensional Brownian motion (W_r, W), the market price of risk vector (θ_r, θ) is unique. It follows (by Theorem 4.9 of Filipovic (2009)) that P^M is the unique EMM for the mixed market, and that the market is complete. As an immediate consequence, any $\mathcal{F}_T^M$-measurable payoff is perfectly replicable by trading in the bank account, the bond and the risky asset. The initial cost of the replicating portfolio is given by the unique no-arbitrage price obtained by taking the P^M-expectation of the discounted payoff.

2.1.3 General notation for the financial market

In Section 3, we will be interested in comparing partial hedging strategies developed in both the bond-only and the mixed market. As will be shown in the next section, the quantities of interest will be functions of the density of the unique EMM, namely Z_T^B in the bond-only market and Z_T^M in the mixed one. In order to simplify the exposition of the results, we will work with a general Z_T of the form

$$Z_T := \exp\left\{ \boldsymbol{\theta}' \boldsymbol{W}(T) - \frac{T}{2} \boldsymbol{\theta}' \boldsymbol{\theta} \right\}, \tag{2.8}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)'$, with $\theta_1, \; \theta_2 \in \mathbb{R}$, and $\boldsymbol{W}(T) = (W_r(T), W(T))'$, and we will denote by P^* the EMM defined by $Z_T = \frac{dP^*}{dP}\big|_{\mathcal{F}_T}$.

To obtain specific results for the bond-only market, it will suffice to let $\boldsymbol{\theta} = (\theta_r, 0)'$, in which case Z_T will be equal to (2.5). For the mixed market, letting $\boldsymbol{\theta} = (\theta_r, -\theta)'$ will yield the desired result. Going forward, we denote

$$\boldsymbol{\theta}^{(B)} = (\theta_r, 0)'$$
$$\boldsymbol{\theta}^{(M)} = (\theta_r, -\theta)'.$$

Therefore, in the rest of this section and in most of Section 3, we will work with general Z_T, $\boldsymbol{\theta}$ and $\boldsymbol{W}$, only to replace them with market-specific values to compare the partial hedging strategies in each market. We also use $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ to denote either $\mathbb{F}^B$ ou $\mathbb{F}^M$, the filtration induced by the financial market.

2.2 Mortality risk

The payoff we seek to hedge in this paper is a simplified version of the payout of a pension plan. As such, it is only paid if the employee survives until retirement. We therefore need to incorporate mortality risk in our framework. To model the time of death of the employee receiving the payout, we borrow the reduced-form framework from credit risk theory (see, for example, Bielecki and Rutkowski (2013)). This will allow us to use the results of Sekine (2000) and of Nakano (2011) on quantile hedging for defaultable securities.

We let τ be a positive random variable denoting the random time of death, with $P(\tau > t) > 0$ for any $0 \leq t \leq T$, and define $\{N_t\}_{0\leq t\leq T}$, with $N_t = \mathbb{1}_{\{\tau\leq t\}}$. We denote by $\mathbb{H} = \{\mathcal{H}_t\}_{0\leq t\leq T}$ the filtration induced by N_t, with

$$\mathcal{H}_t := \sigma\{N_s; 0 \leq s \leq t\},$$

and let $\mathbb{G} := \{\mathcal{G}_t\}_{0\leq t\leq T}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.

We further define the survival process $\{\Lambda_t\}_{0\leq t\leq T}$ with respect to $\mathcal{F}_t$ as

$$\Lambda_t := P(\tau > t|\mathcal{F}_t),$$

and assume that $\Lambda_t > 0$ for $0 \leq t \leq T$. We let the process $\{\lambda_t\}_{0\leq t\leq T}$ be defined implicitly by

$$\Lambda_t = e^{-\int_0^t \lambda_s \, ds}.$$

It follows that λ_t is $\mathcal{F}_t$-adapted. Throughout the paper, we use the following assumption.

Assumption 1. *The survival process Λ_t is independent of the financial market filtration $\mathbb{F}$.*

It follows from Assumption 1 that $\Lambda_t = P(\tau > t)$, for $0 \leq t \leq T$.

We further define

$$M_t = N_t - \int_0^t \lambda_s(1 - N_{s^-})ds, \qquad 0 \leq t \leq T,$$

which can be shown to be a $\mathbb{G}$-martingale (see Bielecki and Rutkowski (2013)). We also let $Z^\delta = \{Z_t^\delta\}_{0\leq t\leq 2T}$ be defined by

$$Z_t^\delta := (1 + \delta_t \mathbb{1}_{\{\tau\leq t\}}) \exp\left(\int_0^{t\wedge T} \delta_s\lambda_s ds\right),$$

where $\delta = \{\delta_t\}_{0 \le t \le 2T}$ is an element of

$$\mathcal{D} = \{\delta : \text{bounded, } \mathbb{G}\text{-predictable, } \delta > -1 \ dt \times dP \ a.e.\}$$

It is possible to show that for $\delta \in \mathcal{D}$, Z^δ satisfies

$$Z_t^\delta = 1 + \int_0^t \delta_s Z_s^\delta dM_s,$$

for $0 \le t \le T$, and is thus a $(\mathbb{G}, P)$-martingale (see for example Bielecki and Rutkowski (2013)). Finally, we have that both $\{Z_t\}_{0 \le t \le T}$, with

$$Z_t = \exp\left\{\boldsymbol{\theta}'\boldsymbol{W}(t) - \frac{t}{2}\boldsymbol{\theta}'\boldsymbol{\theta}\right\},$$

and $\{Z_t Z_t^\delta\}_{0 \le t \le T}$ are $(\mathbb{G}, P)$-positive martingales for any $\delta \in \mathcal{D}$. Furthermore, since Z^δ is orthogonal to any $(\mathbb{F}, P)$-martingales for any $\delta \in \mathcal{D}$, the discounted price processes of the assets traded in the financial market are martingales under the measure P^δ defined by $\frac{dP^\delta}{dP}\big|_{\mathcal{G}_T} = Z_T Z_T^\delta$. It follows that

$$\mathcal{P} := \{P^\delta : \delta \in \mathcal{D}\}$$

defines the class of equivalent martingale measures on $\mathbb{G}$. Note that in particular, $\mathcal{P}$ contains P^*, the unique EMM for the purely financial market defined by the filtration $\mathbb{F}$. This can be shown by letting $\delta \equiv 0$ in (2.2). For more details on this setting, see Bielecki and Rutkowski (2013).

While the purely financial markets described in Section 2.1 were both complete, the addition of mortality renders incomplete the general market combining financial and mortality risk. That is, $\mathcal{G}_T$-measurable payoffs are not necessarily perfectly replicable by trading only in the available assets. For more details on incomplete markets, see for example Karatzas and Shreve (1998).

2.3 Hedging in incomplete markets

A portfolio strategy is characterized by its initial capital V_0 and by a $\mathcal{G}_t$-predictable process $\xi = \{\xi_t\}_{0 \le t \le T}$ that describes the way the portfolio is invested in the different traded assets. A portfolio strategy $\{V_0, \xi\}$ is called admissible if its (discounted) value process[1]

$$V_t = V_0 + \int_0^t \xi_s dX_s$$

satisfies

$$V_t \ge 0, \qquad \text{for all } 0 \le t \le T, \qquad P\text{-a.s.}$$

Here we use $X = \{X_t\}_{0 \le t \le 2T}$ to denote the possibly multi-dimensional discounted price process of the assets traded on the financial market. That is, in the bond-only market, X denotes

[1] Throughout the paper, similarly to Föllmer and Leukert (1999), we consider payoffs and asset prices discounted by the bank account numéraire. This simplifies the notation and does not affect the comparison between the two markets.

the (discounted) price process of the bond with maturity $2T$. In the mixed market, X is two-dimensional and denotes the (discounted) price process of the bond and the risky asset.
It is possible to show (see Nakano (2011)) that the process $\{Z_t Z_t^\delta V_t\}_{0\le t\le T}$ is a supermartingale when $\{V_t\}_{0\le t\le T}$ is the (discounted) value process resulting from an admissible strategy.
We consider a (discounted) $\mathcal{G}_T$-measurable payoff H, with

$$\sup_{\delta\in\mathcal{D}} \mathbb{E}[Z_T Z_T^\delta H] < \infty.$$

Since H is $\mathcal{G}_T$-measurable, it can depend on both the financial market and the survival process, and it is therefore not necessarily possible to replicate it perfectly by trading in the market. A so-called superhedging strategy ensures that $V_T \ge H$, P-a.s. It is the most conservative way to hedge the payoff H. The superhedging cost for the payoff H, denoted by $\Pi(H)$, is defined as

$$\Pi(H) = \inf\{x \ge 0 :\ x + \int_0^T \xi_s \, dX_s \ge H,\ P\text{-a.s. for some admissible strategy } \xi\}.$$

Simple expressions for the superhedging cost of a general claim $H \in \mathcal{G}_T$ do not always exist. However, it is possible to show that for a $\mathcal{G}_T$-measurable payoff H of the form

$$H = Y \mathbb{1}_{\{\tau > T\}}, \tag{2.9}$$

with $Y \in \mathcal{F}_T$, the superhedging strategy for H coincides with the perfect replicating portfolio for Y (see Proposition 2.1 of Nakano (2011)). Since we will use a version of this result throughout the paper, we present it next. Here, $\mathbb{E}^*[\cdot]$ denotes the expectation taken under P^*, the unique EMM in the financial market.

Proposition 2.1 (Proposition 2.1 of Nakano (2011))**.** *Let H be defined as in* (2.9) *and assume* $\mathbb{E}^*[Y] < \infty$. *Then we have*

$$\Pi(H) = \mathbb{E}^*[Y],$$

and the replicating portfolio for Y is the superhedging portfolio for H.

The proof of Proposition 2.1 is given in Nakano (2011). For completeness, we present it in Appendix A.

2.4 Cash balance payoff

In this paper, we consider the payoff of a cash balance pension plan, which depends on the term structure of interest rates and on the survival of an individual. Specifying the form of the claim allows us to obtain analytic expressions for the cost of a quantile hedge. By doing so for each market introduced in Sections 2.1.1 and 2.1.2, we can analyse the impact of the availability of a risky asset on the initial cost of the partial hedge, and thus infer on the incentives created by the quantile hedging criterion in the context of pension plans.
The claim we consider is inspired by a type of hybrid pension plan, and is general enough to cover different types of pension benefits. Nonetheless, the form we use is still restrictive, and

the analysis we perform here could also be modified and applied to other types of interest rate derivatives.

A cash balance plan is similar to a defined contribution plan, but the interest rate credited on the contributions deposited in the employee's account is defined by a pre-determined formula. Cash balance plans are presented in greater details in Hardy, Saunders, and Zhu (2014). The payoff they consider consists of the accumulation up to time T of an initial investment ζ_0 at a guaranteed, pre-determined rate. In this paper, we also take into account the fact that the benefit is paid only if the employee is still alive at T. We however ignore any mortality benefits that could be paid if the plan member does not survive until retirement

Of particular interest is the case where the crediting rate is linked to the term structure of interest rates. Going forward, we follow Hardy, Saunders, and Zhu (2014) and assume that the crediting rate on an initial contribution ζ_0, denoted by $r^c(t;n)$, is given by

$$r^c(t;n) := y_n(t) + g,$$

for $n > 0$ and $0 \leq t \leq T$, where $g \geq 0$ is a constant and $y_n(t)$ is the yield to maturity at time t of a bond with time to maturity n. In our setting, n is usually an integer representing a number of years. For $n > 0$, this yield is defined by

$$y_n(t) := -\frac{1}{n}\log(P(t,t+n)) = \frac{1}{n}(r(t)D(n) - \gamma(n)).$$

The second equality stems from (2.2), and $\gamma(n)$ and $D(n)$ are defined as in (2.3). Note that since $\gamma(n)$ and $D(n)$ only depend on n, the yield rate $y_n(t)$ only depends on t through $r(t)$.

We also wish to consider constant crediting rates. To denote this case, we write $r^c(t;0) = g$. Therefore, the case $r^c(t;n) = g$ will be denoted by $n = 0$.

Examples of such crediting rates include a pension plan that credits the current 3-year yield rate, plus 0.5%, or the current 10-year yield rate. Letting $n = 0$ leads to the case where a constant rate is guaranteed.

The payoff we consider is thus the accumulation at rate r^c of the initial investment ζ_0. We denote the discounted financial payoff by ζ. It represents the amount paid given that the employee survives to T, and it is given by:

$$\begin{aligned}\zeta &= \zeta_0 e^{\int_0^T (r^c(t;n) - r(t))dt} \\ &= \begin{cases} \zeta_0 e^{gT + \int_0^T (y_n(t) - r(t))dt}, & n > 0, \\ \zeta_0 e^{gT - \int_0^T r(t)dt}, & n = 0. \end{cases}\end{aligned} \tag{2.10}$$

Without loss of generality, we let $\zeta_0 = 1$ from now on.

Since the plan pays out only if the plan member survives to T, the (discounted) payoff we seek to hedge, denoted by H_T, is defined as

$$H_T := \zeta \mathbb{1}_{\{\tau > T\}} = \zeta(1 - N_T). \tag{2.11}$$

Remark 2. Our definition of H_T simplifies the total payoff of a cash balance plan in different ways. First, it does not take into account ongoing contributions made to the account. These future contributions can be treated separately in a similar manner, or they can be included in the initial amount at the next valuation date. Second, as in Hardy, Saunders, and Zhu (2014), we ignore any benefits that could be paid should the plan member die before T.

Remark 3. Note also that by letting $n = g = 0$ and by using τ to denote a default time, H_T is the discounted payoff of a defaultable zero-coupon bond with maturity T. Quantile hedging a defaultable zero-coupon bond is thus a special case of the results that follow. Our framework can also be used to model the payoff of a defined benefit plan, whose payout at T is known (under certain assumptions on salary growth and mortality) at time 0. Therefore, our analysis can be extended beyond cash balance plans.

Note that the pure financial payoff ζ is both $\mathcal{F}_T$- and $\mathcal{G}_T$-measurable, while the payoff H_T is only $\mathcal{G}_T$-measurable. Since the pure financial market is complete, ζ can be perfectly replicated by trading in the available assets. This is however not the case for H_T. We have the following results on the superhedging cost of H_T.

Corollary 2.2. *Let H_T be as in* (2.11)*. Then, we have*

$$\Pi(H_T) = \mathbb{E}^*[\zeta], \tag{2.12}$$

where $\mathbb{E}^*[\cdot]$ denotes the expectation taken under the P^*-measure. Furthermore, the replicating portfolio for ζ is the superhedging portfolio for H_T.

Proof. Corollary 2.2 follows immediately from Proposition 2.1. □

It our market model, it is possible to obtain the density of $Z_T\zeta$ under P and to derive an analytic expression for $\mathbb{E}^*[\zeta]$. We present those results here, since they will be used throughout the rest of the paper in the context of quantile hedging.

Proposition 2.3. *Let ζ be the payoff defined by* (2.10) *and let*

$$Z_T = \exp\left\{\boldsymbol{\theta}'\boldsymbol{W}(T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta}\right\}.$$

Then, under P, $Z_T\zeta$ follows a lognormal distribution with parameters $\left(\eta(n,T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta}, \vartheta^2(n,T;\boldsymbol{\theta})\right)$, where

$$\eta(n,T) = \begin{cases} gT - \frac{\gamma(n)T}{n} + \widetilde{D}(n)\left((r_0 - b)\frac{1-e^{-aT}}{a} + bT\right), & n > 0, \\ gT - \left((r_0 - b)\frac{1-e^{-aT}}{a} + bT\right), & n = 0, \end{cases}$$

$$\begin{aligned} \vartheta^2(n,T;\boldsymbol{\theta}) = {} & \frac{\widetilde{D}^2(n)\sigma_r^2}{a^2}\left(T - D(T) - \frac{a}{2}D^2(0,T)\right) \\ & - \frac{2\theta_r\sigma_r\widetilde{D}(n)}{a}\left(D(T) - T\right) + \boldsymbol{\theta}'\boldsymbol{\theta}T, \end{aligned}$$

and

$$\widetilde{D}(n) = \begin{cases} \frac{D(n)-n}{n}, & n > 0 \\ -1, & n = 0. \end{cases}$$

The proof of Proposition 2.3 is given in Appendix B. Proposition 2.3 allows us to calculate $\mathbb{E}^*[\zeta]$, which is presented next.

Corollary 2.4. *Let ζ and Z_T be defined as in Proposition 2.3. $\mathbb{E}^*[\zeta]$ is then given by*

$$\mathbb{E}^*[\zeta] = e^{\eta(n,T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta} + \frac{1}{2}\vartheta^2(n,T;\boldsymbol{\theta})}.$$

Proof. Note that $\mathbb{E}^*[\zeta] = \mathbb{E}[Z_T\zeta]$. The result follows directly from the distribution of $Z_T\zeta$ given in Proposition 2.3. □ □

Remark 4. The unique no-arbitrage price of the payoff ζ is the same in both the bond-only market and the mixed market. Heuristically, this is due to the fact that the perfect replicating strategy that underlies the no-arbitrage price only involves investing in bonds, whether or not stocks are available. Mathematically, using the independence of $W(t)$ and $W_r(t)$, we have

$$\begin{aligned}
\mathbb{E}\left[Z_T^M\zeta\right] &= \mathbb{E}\left[e^{-\theta W(T) - \frac{1}{2}\theta^2 T} Z_T^B \zeta\right] \\
&= \mathbb{E}\left[e^{-\theta W(T) - \frac{1}{2}\theta^2 T}\right] \mathbb{E}\left[Z_T^B\zeta\right] \\
&= \mathbb{E}\left[Z_T^B\zeta\right].
\end{aligned}$$

3 Quantile hedging the cash balance payoff

Corollary 2.2 shows that it is possible to build a superhedging portfolio that can cover the claim H_T with probability 1. The initial cost of such a strategy is the unique no-arbitrage price of the financial claim ζ. For different reasons, a pension plan sponsor may not be willing to invest the full market value of ζ in a hedging strategy, and is therefore exposed to the risk that the value at T of the portfolio is not sufficient to cover the pension payout. In this context, for a given initial portfolio value, what is the probability that the final value of the strategy is at least equal to the payout? What is the cost of a hedging strategy that covers the payout with a given probability? Are these quantities different when investment in equity is allowed? In this section, we derive analytical results that allow us to compare the cost and effectiveness of such quantile hedging strategies in both markets presented in Section 2.1.
More precisely, we consider an investor (or plan sponsor) who wants to set up a self-financing hedging portfolio to that will maximize the probability of V_T, the final value of the trading strategy, being at least equal to H_T, given that the initial cost V_0 is at most a given threshold $\widetilde{V}_0$. This corresponds to finding the admissible strategy $\{V_0, \xi\}$ that maximizes

$$P[V_T \geq H_T],$$

under the constraint $V_0 \leq \widetilde{V}_0$, where $\widetilde{V}_0 < \Pi(H_T)$. We refer to this optimization problem as **(OP1)**.
We are also interested in the case where a hedger wants to find the optimal hedging strategy that will cover the claim H_T with a probability of at least $1 - \varepsilon$, at a minimal cost. Therefore,

we are looking for the admissible strategy $\{V_0, \xi\}$ that minimizes V_0 under the constraint

$$P\left[V_0 + \int_0^T \xi_s \, dX_s \geq H_T\right] \geq 1 - \varepsilon.$$

If the initial cost of the strategy is equal to $\Pi(H_T)$, then superhedging is possible. Thus, we expect that the value $\widetilde{V}_0$ that solves the minimization will be strictly less than $\Pi(H_T)$ for $\varepsilon > 0$. We denote this optimization problem **(OP2)**.

Such hedging strategies were considered in a general semi-martingale settings in Föllmer and Leukert (1999). In this section, we apply the results of Sekine (2000) and Nakano (2011) on quantile hedging of defaultable claims, in order to study the resulting investment incentives related the pension payoff presented in Section 2.4. We first provide results using the "general market" notation introduced in Section 2.1.3. Market-specific notation will then be used to analyse the impact of the availability of a risky asset on the initial cost of the hedge. We also assess the riskiness of the resulting hedge.

Throughout the section, we mostly focus on minimizing the initial cost **(OP2)**, since we believe it better represents the preoccupations of a pension plan sponsor. Nonetheless, we also present results on the maximal probability of success for a given initial cost (**OP1**).

3.1 Minimizing the initial cost of the hedging strategy

We first consider **(OP2)**, that is, we seek to minimize the cost of a partial hedging strategy, given that the pension payout is covered by the hedging portfolio with a given probability.

Theorem 3.1 (Theorem 2(A) of Sekine (2000))**.** *Assume that there exists $k^* = k^*(\varepsilon)$ satisfying*

$$\mathbb{E}[\mathbb{1}_{\{A_1(k^*)\}}(1 - \Lambda_T) + \mathbb{1}_{A_2(k^*)}] = 1 - \varepsilon,$$

with $A_1(k) = \{\Lambda_T \leq k Z_T \zeta\}$ and $A_2(k) = \{\Lambda_T > k Z_T \zeta\}$. Then the superhedging strategy of the claim

$$\widetilde{H}_T = \mathbb{1}_{A_2(k^*)} \zeta (1 - N_T)$$

*is a solution of **(OP2)**.*

Proof. Theorem 3.1 is a special case of Theorem 2(A) of Sekine (2000), which pertains to a defaultable claim with partial recovery d if default occurs before T. We obtain Theorem 3.1 by setting $d = 0$. □

By Theorem 3.1, the initial cost of the quantile hedging strategy is given by the superhedging price of the modified payoff $\mathbb{1}_{A_2(k^*)} \zeta (1 - N_T)$. We give an explicit expression for this price in the following proposition. Throughout the paper, $\Phi(\cdot)$ denotes the standard normal distribution function and $z_\alpha = \Phi^{-1}(\alpha)$ is the α-quantile of the standard Normal distribution.

Proposition 3.2. *Let H_T and Z_T be defined by* (2.11) *and* (2.8)*, respectively, and let $\{V_t\}_{0 \leq t \leq T}$ denote the value process of an admissible hedging strategy. The minimal cost V_0 of the quantile hedging strategy that satisfies*

$$P\left[V_T \geq H_T\right] \geq 1 - \varepsilon$$

is given by

$$\mathbb{E}^*\left[\zeta \mathbb{1}_{A_2(k^*)}\right] = \mathbb{E}^*\left[\zeta\right] \Phi\left(z_{1-\varepsilon^\star} - \vartheta(n,T;\boldsymbol{\theta})\right), \tag{3.1}$$

where $\vartheta^2(n,T;\boldsymbol{\theta})$ *is as defined in Proposition 2.3 and* $\varepsilon^\star = \frac{\varepsilon}{\Lambda_T}$.

Proof. This result is an application of Theorem 3.1. The minimal cost of the strategy corresponds to the superhedging price of the modified payoff $\widetilde{H}_T = \mathbb{1}_{A_2(k^*)}\zeta(1-N_T)$.
To solve for k^*, observe that

$$\mathbb{E}[\mathbb{1}_{A_1(k)}(1-\Lambda_T) + \mathbb{1}_{A_2(k)}] = (1-\Lambda_T) + P(\zeta Z_T < k\Lambda_T)\Lambda_T.$$

From the distribution of ζZ_T, it is clear that k^* exists. Using Proposition 2.3, we obtain

$$k^* = \Lambda_T \exp\left\{-(\eta(n,T) - \frac{T}{2})\boldsymbol{\theta}'\boldsymbol{\theta} - z_{1-\varepsilon^\star}\vartheta(n,T;\boldsymbol{\theta})\right\}, \tag{3.2}$$

with $\varepsilon^\star = \frac{\varepsilon}{\Lambda_T}$. It follows that

$$A_2(k^*) = \left\{Z_T\zeta < \exp\left\{\eta(n,T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta} + z_{1-\varepsilon^\star}\vartheta(n,T;\boldsymbol{\theta})\right\}\right\},$$

and thus $\mathbb{1}_{A_2(k^*)}\zeta \in \mathcal{F}_T$. By Proposition 2.1, the superhedging portfolio for $\widetilde{H}_T$ is the replicating portfolio for $\mathbb{1}_{A_2(k^*)}\zeta$, so that the minimal cost V_0 is given by

$$\mathbb{E}^*[\zeta\mathbb{1}_{A_2(k^*)}] = \mathbb{E}\left[Z_T\zeta\mathbb{1}_{\left\{Z_T\zeta<\exp\left(\eta(n,T)-\frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta}+z_{1-\varepsilon^\star}\vartheta(n,T;\boldsymbol{\theta})\right)\right\}}\right].$$

The result follows from the distribution of $Z_T\zeta$. □ □

As expected, the initial cost of the quantile hedging strategy given in Proposition 3.2 is bounded above by $\mathbb{E}^*[\zeta]$, the superhedging price of the original payoff H_T.

3.1.1 Comparison of the initial hedging cost in the bond-only and the mixed market

Proposition 3.2 gives an explicit expression for the minimal cost of a quantile hedging strategy with probability of success $1-\varepsilon$ using the general financial market notation introduced in Section 2.1.3. Therefore, this result applies to both the bond-only and the mixed market; to get market-specific expressions, it suffices to replace the general $\boldsymbol{\theta}$ by $\boldsymbol{\theta}^{(B)}$ or $\boldsymbol{\theta}^{(M)}$ appropriately. This allows us to compare the cost of the optimal quantile hedging strategy in the two markets presented in Section 2.1.
Using market-specific notation, we have

$$\mathbb{E}^B\left[\zeta\mathbb{1}_{A_2(k^*;B)}\right] = \mathbb{E}^B\left[\zeta\right]\Phi(z_{1-\varepsilon^\star} - \vartheta(n,T;\boldsymbol{\theta}^{(B)})) \tag{3.3}$$

and

$$\mathbb{E}^M\left[\zeta\mathbb{1}_{A_2(k^*;M)}\right] = \mathbb{E}^M\left[\zeta\right]\Phi(z_{1-\varepsilon^\star} - \vartheta(n,T;\boldsymbol{\theta}^{(M)})), \tag{3.4}$$

where $\mathbb{E}^M[\cdot]$ and $\mathbb{E}^B[\cdot]$ denote the expectations taken under the measures P^B and P^M, respectively.

As explained in Remark 4, the unique no-arbitrage price of the payoff ζ is the same in both financial markets, that is, $\mathbb{E}^B[\zeta] = \mathbb{E}^M[\zeta]$. Therefore, being able to invest in equity only affects the initial cost of the hedging strategy through the P^*-variance parameters of the original payoff, $\vartheta^2(n,T;\boldsymbol{\theta}^{(B)})$ and $\vartheta^2(n,T;\boldsymbol{\theta}^{(M)})$. From the definition of $\vartheta^2(n,T;\boldsymbol{\theta})$, we have

$$\begin{aligned}\vartheta^2(n,T;\boldsymbol{\theta}^{(M)}) &= \vartheta^2(n,T;\boldsymbol{\theta}^{(B)}) + \theta^2 T \\ &> \vartheta^2(n,T;\boldsymbol{\theta}^{(B)}) \end{aligned} \tag{3.5}$$

when $\theta \neq 0$ (that is, when the market price of equity risk differs from 0). In that case, $\vartheta(n,T;\boldsymbol{\theta}^{(M)}) > \vartheta(n,T;\boldsymbol{\theta}^{(B)})$. The monotonicity of $\Phi(\cdot)$ simplifies the comparison between the markets. We present this result in Corollary 3.3.

Corollary 3.3. *Let $\mathbb{E}^M[\cdot]$ and $\mathbb{E}^B[\cdot]$ denote the expectations taken under the measures P^B and P^M, respectively, and let*

$$A_2(k^*;B) = \left\{ Z_T\zeta < e^{\eta(n,T)-\frac{T}{2}(\boldsymbol{\theta}^{(B)})'\boldsymbol{\theta}^{(B)}+z_{1-\varepsilon^\star}\vartheta(n,T;\boldsymbol{\theta}^{(B)})} \right\}$$

$$A_2(k^*;M) = \left\{ Z_T\zeta < e^{\eta(n,T)-\frac{T}{2}(\boldsymbol{\theta}^{(M)})'\boldsymbol{\theta}^{(M)}+z_{1-\varepsilon^\star}\vartheta(n,T;\boldsymbol{\theta}^{(M)})} \right\}.$$

If the market price of equity risk $\theta \neq 0$, then

$$\mathbb{E}^M\left[\zeta \mathbb{1}_{A_2(k^*;M)}\right] < \mathbb{E}^B\left[\zeta \mathbb{1}_{A_2(k^*;M)}\right],$$

where ζ is the payoff defined in (2.10)*.*

Corollary 3.3 highlights the fact that when the market price of equity risk is different from zero, being able to invest in the equity market can reduce the initial cost of a self-financing quantile hedging strategy.

The application of Proposition 3.2 to each market also gives further insight into the characteristics of the mixed market that influence the cost of the hedging strategy. In particular, σ_2, which characterizes the dependence between the randomness of the stock price and the randomness of the interest rate, does not appear directly in (3.1). However, the market price of equity risk θ, which appears only in $\boldsymbol{\theta}^{(M)}$, is linked to the ratio of the drift of the risky asset to its volatility, both of which are functions of σ_2. Therefore, the dependence between the risky asset and the interest rate will impact the cost of the hedge through the market price of equity risk. The availability of the risky asset reduces the cost of the hedging strategy in comparison to a bond-only hedge, as long as the market price of equity risk is different from 0. The dependence between the risky asset and the bond should also impact the quantile hedging strategy.

Expressing the cost of the quantile hedge obtained in Proposition 3.2 using the parameters specific to each market, as it is done in the proof of Corollary 3.3, also highlights the impact of the maturity T on the hedging cost. From (3.3) and (3.4), it is clear that the only difference in the hedging cost comes from the P^*-variance of the original payoff. Then, writing $\vartheta^2(n,T;\boldsymbol{\theta}^{(M)})$ in terms of $\vartheta^2(n,T;\boldsymbol{\theta}^{(B)})$ as in (3.5) shows that as the maturity of the payoff T increases, the difference between the initial hedging cost in the bond-only and the mixed market grows. This indicates that the equity risk premium has a more significant impact over a longer period of time.

Finally, one would expect the cost of the hedging portfolio to increase to the superhedging price of the payoff H_T, given in (2.12), as ε goes to 0, regardless of the available assets. Since $\lim_{x\to\infty}\Phi(x)=1$, (3.3) and (3.4) satisfy this requirement. In Section 4, these last observations are illustrated with a numerical example.

3.1.2 Unhedged payoff

The quantile hedge whose cost was derived in the previous section is only a partial hedge. On the set $A_2^c(k^*)\cap\{\tau>T\}$, where $A_2^c(k^*)$ is the complement of $A(k^*)$, the payoff is not replicated, and the hedger incurs a loss. We derive here the expected value of this loss, given that it is strictly positive.

For a given probability $1-\varepsilon$ of a successful hedge, we showed in the previous section that the availability of equity can reduce the price of the replicating portfolio for the modified claim $\zeta\mathbb{1}_{A_2(k^*)}$. We are now interested in the average loss incurred by the pension plan sponsor given that the hedge fails.

As in Section 3.1.1, we first derive an expression using the general financial market notation; that is, we use the Radon-Nikodym derivative given by (2.8). The result thus obtained will then apply to both the bond-only and the mixed market. To obtain market-specific expressions, it will suffice to replace $\boldsymbol{\theta}$ by $\boldsymbol{\theta}^{(B)}$ or $\boldsymbol{\theta}^{(M)}$ appropriately.

Using the general financial market notation, we define the loss at maturity L by $L:=H_T-\widetilde{V}_T$, where $\widetilde{V}_T$ is the value at T of the superhedging portfolio for the modified payoff $\widetilde{H}_T=\mathbb{1}_{A_2(k^*)}\zeta(1-N_T)$ as in Theorem 3.1.[2] Therefore, the loss is equal to the full financial payoff ζ on $A_2^c(k^*)\cap\{\tau>T\}$. The next proposition gives an expression for its P-expected value, given that the loss is positive. It is the (discounted) amount that the investor can expect to lose on average when a loss occurs.

Proposition 3.4. *Let $\widetilde{V}_T$ be the value at time T of the superhedging portfolio for $\widetilde{H}_T$ and let $L=H_T-\widetilde{V}_T$ be the loss at maturity resulting from a quantile hedge that has a probability of success of at least $1-\varepsilon$. The expected value of L conditional on $L>0$ is given by*

$$\mathbb{E}[L|L>0]=(\varepsilon^\star)^{-1}e^{\eta(n,T)+\frac{1}{2}\nu^2 s_I^2}\Phi\left(\frac{\nu^2 s_I^2+\nu\mu_{I|W}\theta_r T}{\vartheta(n,T;\boldsymbol{\theta})}-z_{1-\varepsilon^\star}\right), \tag{3.6}$$

where $\varepsilon^\star=\frac{\varepsilon}{\Lambda_T}$, $\nu=\widetilde{D}(n)\sigma_r e^{-aT}$ and $s_I^2=s_{I|W}^2+\mu_{I|W}T$, with

$$\begin{aligned}\mu_{I|W}&=-\frac{e^{aT}}{aT}(D(T)-T)\\ s_{I|W}^2&=\frac{e^{2aT}}{a^2}\left(D(T)-D^2(0,T)\left(\frac{a}{2}+\frac{1}{T}\right)\right).\end{aligned}$$

[2] Throughout this work, we consider payoffs discounted with the bank account numéraire, so that the quantity we are studying here is in fact the discounted loss.

The proof of Proposition 3.4 is given in Appendix C.

Proposition 3.4 shows that $\mathbb{E}[L|L>0]$ only depends on the availability of the risky asset through the P^*-standard deviation $\vartheta(n,T;\boldsymbol{\theta})$. In order to compare the loss resulting from the unhedged part of the payoff in each market, we replace the general market price of risk vector $\boldsymbol{\theta}$ in (3.6) by a specific one, $\boldsymbol{\theta}^{(B)}$ or $\boldsymbol{\theta}^{(M)}$. We also let the subscripts (B) and (M) identify the market in which the quantile hedging strategy is derived.

Expressing the result of Proposition 3 for each specific market gives

$$\mathbb{E}[L^{(B)}|L^{(B)}>0] = (\varepsilon^\star)^{-1} e^{\eta(n,T)+\frac{1}{2}\nu^2 s_I^2} \Phi\left(\frac{\nu^2 s_I^2 + \nu\mu_{I|W}\theta_r T}{\vartheta(n,T;\boldsymbol{\theta}^{(B)})} - z_{1-\varepsilon^\star}\right) \tag{3.7}$$

and

$$\mathbb{E}[L^{(M)}|L^{(M)}>0] = (\varepsilon^\star)^{-1} e^{\eta(n,T)+\frac{1}{2}\nu^2 s_I^2} \Phi\left(\frac{\nu^2 s_I^2 + \nu\mu_{I|W}\theta_r T}{\vartheta(n,T;\boldsymbol{\theta}^{(M)})} - z_{1-\varepsilon^\star}\right). \tag{3.8}$$

From (3.5), we know that $\vartheta(n,T;\boldsymbol{\theta}^{(M)}) > \vartheta(n,T;\boldsymbol{\theta}^{(B)})$ when $\theta \neq 0$. Therefore, the impact of the availability of the equity depends on the sign of the numerator $\nu^2 s_I^2 + \nu\mu_{I|W}\theta_r T$ in (3.7) and (3.8). This is formalized in the following corollary.

Corollary 3.5. *Let $\mathbb{E}[L^{(B)}|L>0]$ and $\mathbb{E}[L^{(M)}|L>0]$ be the expected conditional loss in the bond-only and the mixed market, respectively, as given (3.7) and (3.8). Then, we have*

$$\begin{cases} \mathbb{E}[L^{(M)}|L^{(M)}>0] < \mathbb{E}[L^{(B)}|L^{(B)}>0], & \text{if } \theta_r < \kappa, \\ \mathbb{E}[L^{(M)}|L^{(M)}>0] \geq \mathbb{E}[L^{(B)}|L^{(B)}>0], & \text{if } \theta_r \geq \kappa, \end{cases}$$

where $\kappa = -\frac{\nu s_I^2}{\mu_{I|W} T}$.

Proof. The proof follows from the definition of ν, s_I^2 and $\mu_{I|W}$ in Proposition 3.4. Note that it is possible to show that $\nu < 0$ for all $a, T > 0$ and $n \geq 0$. □ □

The previous result is presented in terms of θ_r for ease of interpretation. The idea behind Corollary 3.5 is that when the market price of interest rate risk is too low, having the opportunity to invest in the risky asset can reduce the risk (expressed in terms of the unhedged loss) of the strategy. In Section 5, we explore this result further through numerical examples.

The next proposition qualifies the behaviour of the expected loss as the probability of loss ε goes to 0. We use the general notation L to refer to the loss associated with quantile hedging in either market, $L^{(B)}$ or $L^{(M)}$, since the result is the same in both cases.

Proposition 3.6. *Let $L = H_T - \widetilde{V}_T$, and denote $\kappa = -\frac{\nu s_I^2}{\mu_{I|W} T}$. Then,*

$$\lim_{\varepsilon \to 0} \mathbb{E}[L|L>0] = \begin{cases} \infty, & \theta_r < \kappa, \\ \Lambda_T e^{\eta(n,T)+\frac{1}{2}\nu^2 s_I^2}, & \theta_r = \kappa, \\ 0, & \theta_r > \kappa. \end{cases}$$

Proof. The proof follows from Lemma D.1, presented in Appendix D. □ □

We showed in the previous section that as ε decreases to 0, the price of the hedging strategy approaches the superhedging price of the payoff. Intuitively, one may imagine that as the hedging strategy approaches perfect replication, $\mathbb{E}[L|L>0]$ would go to 0, whether investing in equity is permitted or not. However, this is not necessarily true, as shown in Proposition 3.6. In fact, the expectation of the loss only goes to 0 when $\nu^2 s_I^2 + \nu\mu_{I|W}\theta_r T$ is negative, that is, when $\theta_r > \kappa$. In Section 5, we present numerical examples in the case where this condition is not satisfied. Proposition 3.4 highlights one of the risks linked to the use of the probability of loss as a hedging criterion under such market conditions.

The positive probability of an unbounded loss resulting from quantile hedging strategies for certain parameter sets has already been discussed in the literature, in particular in the introduction of Barski (2016). It is one of the pitfalls of classical quantile hedging and represents a reason why the probability of a loss is not necessarily an appropriate hedging criterion, especially for pension payoffs. This shortcoming of quantile hedging has motivated further work on partial hedging, for example in Barski (2016); Föllmer and Leukert (2000). The application of these results to our setting is out of the scope of this paper.

One important remark is that the limiting behaviour of the expected loss does not depend on the market (bonds-only or mixed) in which the claim is hedged. As previously explained, Proposition 3.6 uses the general notation L because it holds for the loss in the bond-only market $L^{(B)}$ as well as for the one in the mixed market $L^{(M)}$. One can thus conclude that if the expected loss is unbounded when the partial hedging portfolio is partially invested in equity, it has the same limiting behaviour in the bond-only market. While the size of the loss might differ, unboundedness of the expected loss as ε approaches 0 is therefore only a result of the structure of the bond part of the market.

3.2 Maximizing the probability of success

The first part of Section 3 focuses on the minimal hedging cost for a given probability of success, because we believe that it better represents the problem faced by a pension plan sponsor. In fact, it seems reasonable to set a maximal probability of financial loss, and to fund the plan accordingly. Nonetheless, we also present here an explicit expression for the maximal probability of success of a quantile hedge given an upper bound on its initial cost. This is the problem we refer to as **(OP1)**.

Theorem 3.7 (Theorem 1(A) of Sekine (2000))**.** *Assume that there exists $k^\star = k^\star(\widetilde{V}_0)$ satisfying*

$$\mathbb{E}^*[\zeta \mathbb{1}_{A(k^\star)}] = \widetilde{V}_0,$$

*with $A(k) = \{\Lambda_T > kZ_T\zeta\}$. Then the superhedging strategy of the modified claim $\widetilde{H}_T = \mathbb{1}_{A(k^\star)}\zeta(1-N_T)$ is a solution of **(OP1)**.*

Proof. Theorem 3.7 is a special case of Theorem 1(A) of Sekine (2000), which studies defaultable claims with partial recovery d. Theorem 3.7 is obtained by letting $d = 0$. □

Below, we present an explicit expression for the maximal probability of success of the quantile hedge subject to the initial cost being at most $\widetilde{V}_0$.

Proposition 3.8. *Let H_T and Z_T be defined by* (2.11) *and* (2.8)*, respectively, and let $\{V_t\}_{0\leq t\leq T}$ denote the value process of an admissible hedging strategy. The maximal probability $P(V_T \geq H_T)$ subject to $V_0 \leq \widetilde{V}_0$ is equal to*

$$(1-\Lambda_T) + \Lambda_T \Phi\left(\Phi^{-1}\left(\frac{\widetilde{V}_0}{\mathbb{E}^*[\zeta]}\right) + \vartheta(n,T;\boldsymbol{\theta})\right).$$

Proof. This result is an application of Theorem 3.7. The optimal set $A(k^\star)$ is defined by $\mathbb{E}^*[\mathbb{1}_{A(k^\star)}] = \widetilde{V}_0$, which can be re-written as

$$\mathbb{E}\left[Z_T\zeta\mathbb{1}_{\{Z_T\zeta<\Lambda_T k^{-1}\}}\right] = \widetilde{V}_0.$$

Using the distribution of $Z_T\zeta$ given in Lemma 2.3 yields $k^\star$. The optimal strategy is thus given by the superhedging portfolio for $\mathbb{1}_{A(k^\star)}\zeta(1-N_T)$, which coincides with the replicating strategy for $\mathbb{1}_{A(k^\star)}\zeta$, by Proposition 2.1. Using the independence between the survival process and the financial market, we can write the probability of success as

$$\begin{aligned} P(V_T \geq H_T) &= P(\mathbb{1}_{A(k^\star)}\zeta \geq H_T) \\ &= (1-\Lambda_T) + \Lambda_T P(A(k^\star)). \end{aligned}$$

The result follows from the distribution of $Z_T\zeta$. □ □

4 Numerical Illustrations

In this section we complement the analysis of the results of Propositions 3.2 and 3.6 with some numerical illustrations.
For this purpose, we use the following parameters:

Table 1: Market Parameters.

Interest rate	Equity
$a = 0.035$	$\sigma_1 = 0.18$
$b = 0.02$	$\sigma_2 = 0.05$
$\sigma_r = 0.008$	$\theta = 0.24$
$r_0 = 0.02$	
$\theta_r = 0.12$	

Our market model is too simple to provide a good fit to market data. Nonetheless, we choose a parameter set that is representative of typical market conditions. Using the interest rate parameters of Table 1, the long-term unconditional mean for $r(t)$ is 0.02, while its long term unconditional standard deviation is 0.03024. These values are in line with those obtained by Hardy, Saunders, and Zhu (2014). The equity parameters yield log-returns with a standard deviation of 0.1868. The drift of the log-return on equity over the risk-free rate is 0.03175.

We consider a cash balance payoff with maturity $T = 20$ years, unless otherwise stated, where the constant part of the crediting rate is equal to $g = 0.01$ and the variable part is linked to the 10-year spot rate (such that $n = 10$). We still assume that $\zeta_0 = 1$.
For the probability of survival to maturity $\Lambda_T = P(\tau > T)$, we use the 2014 Canadian Pensioner's Mortality Table[3] (CMP2014) for a female pensioner, without mortality improvements. We consider that T represents the time at which the plan member reaches 65. Therefore, for $T = 20$, we use the probability that a woman aged 45 at $t = 0$ is still alive at 65 years old. In that case, we have $P(\tau > 20) \approx 0.9539$. In the examples where T varies, we vary the age of the plan member at $t = 0$ accordingly.

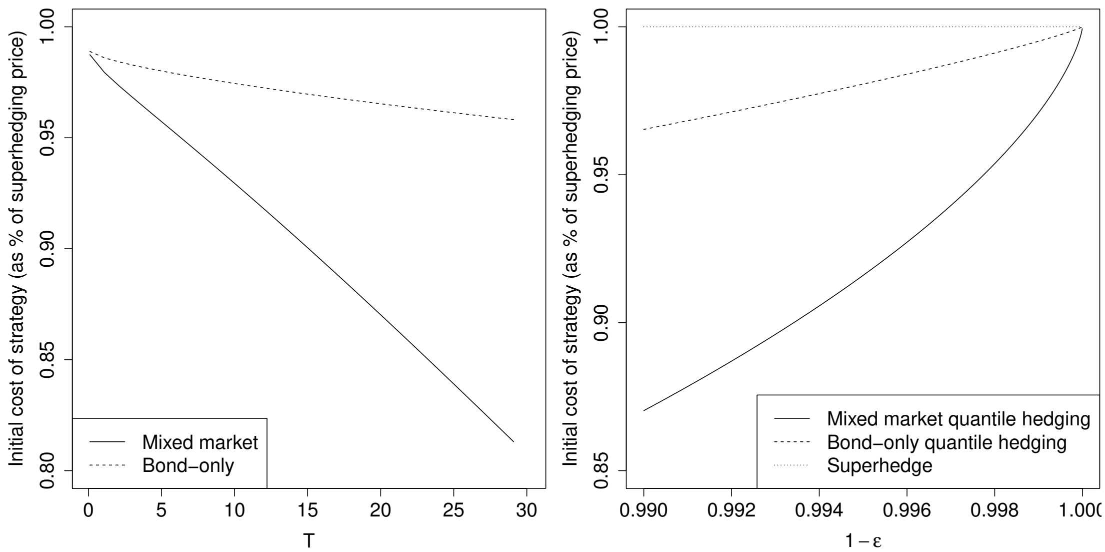
4.1 Initial cost of the hedging strategy

Figure 1a confirms the analysis of Section 3.1.1 concerning the cost of the hedging strategy as T increases. The presence of equity in the market has a more significant impact when the maturity of the payoff is longer. Figure 1b shows that, as expected, the cost of the hedging strategy increases to $\mathbb{E}^*[\zeta]$ as ε goes to 0, whether equity is present in the market or not. It also is interesting to note that with the parameters we use, the initial cost of the strategy in the mixed market remains significantly lower than its bond-only counterpart, even for small values of ε.
Figure 1c indicates that the initial cost of the strategy is slightly more sensitive to the probability of survival when a risky asset is available. Nonetheless, for $P(\tau > T) \geq 0.8$, the probability of survival does not have an important impact on the initial cost. Note that we only tested values above 0.8, which corresponds to the probability of a 55 year old woman surviving 25 years, according to the CPM2014.
Further numerical analysis shows that the initial cost of the quantile hedging strategy has a similar behaviour for various values of g and n, including the zero-coupon bond case $n = 0$. Therefore, we do not illustrate these results here.

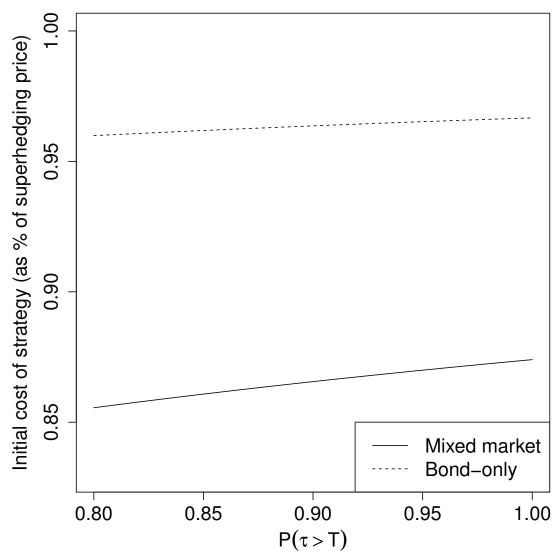
4.1.1 Unhedged payoff

As explained in Section 3.1.2, the impact of equity on the size and the limiting behaviour of the expected loss depends on the size of the market price of interest rate risk θ_r. Using the parameters in Table 1, we have $\kappa = 0.01269 < \theta_r$. Therefore, the presence of equity leads to an increase in the expected loss. In both cases, the expected loss approaches 0 as $\varepsilon \to 0$.
Figure 2 presents the sensitivity of the expected loss to the market price of equity risk θ, as well as its asymptotic behaviour when ε goes to 0. We illustrate the expected loss as a percentage of the superhedging price of the claim, for comparison purposes. While a higher market price of equity risk leads to a lower initial hedging cost, it is also associated with a higher expected loss, if loss occurs. One can consider this higher expected loss as the price to pay for a less expensive hedge. It also points out a flaw in the optimization criteria. In fact, by focusing only on the

[3]The 2014 Canadian Pensioner's Mortality Table is available at `http://www.cia-ica.ca/docs/default-source/2014/214013e.pdf`.



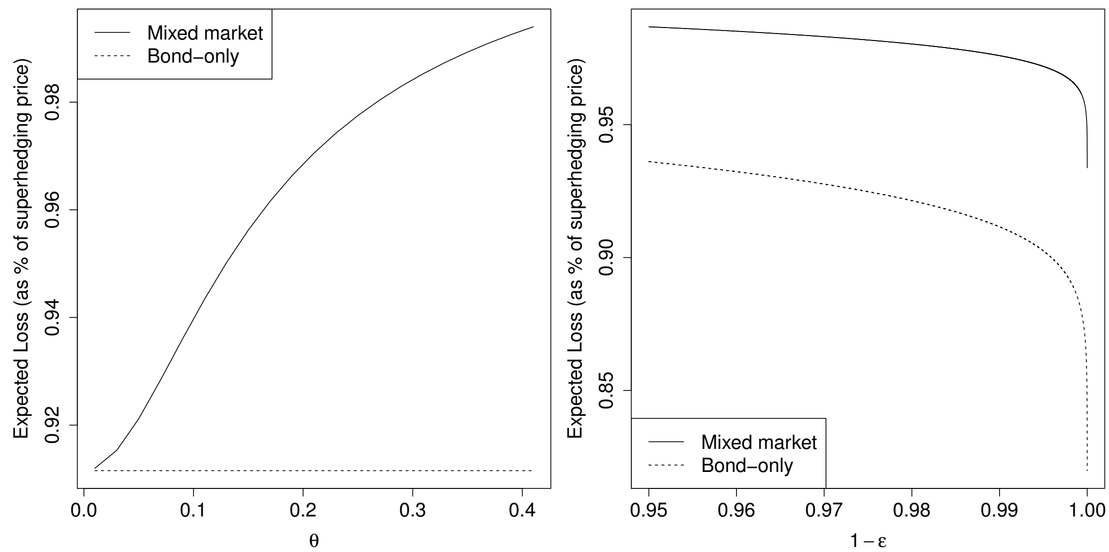
(a) $\varepsilon = 0.01$ (b) $T = 20,\ P(\tau > T) = 0.9539$



(c) $\varepsilon = 0.01,\ T = 20$

Figure 1: Sensitivity of the initial cost of the strategy to (a) maturity T of the payoff, (b) probability of loss ε and (c) survival probability $P(\tau > T)$.

probability of success, the optimal hedging strategy ignores the size of the loss that occurs with probability ε. Not controlling the size of this loss can represent an important risk for the hedger. The behaviour of $\mathbb{E}[L|L > 0] \to 0$ as $\varepsilon \to 0$ is illustrated, albeit not perfectly, in Figure 2b. Although the limit of the expected loss is 0 in both markets, convergence is very slow and hard to illustrate numerically. This especially true in the mixed market model, where the expected loss stays above 80% of the superhedging price of the payoff for values of ε as small as 0.005.
To explore the behaviour of the expected loss when $\theta_r < \kappa$, we also let $\theta_r = 0.01$. This has two effects: first, the expected loss is now lower in the mixed market than in its bond-only counterpart, and second, the expected loss increases as the probability of success $1-\varepsilon$ approaches

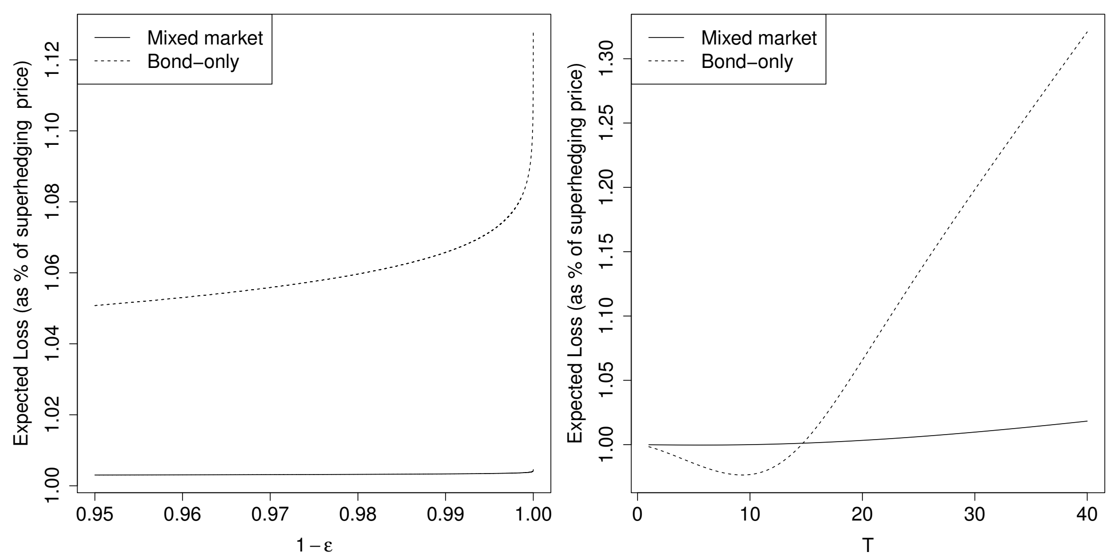


(a) $\varepsilon = 0.01$ (b) $\theta = 0.24$

Figure 2: Sensitivity of $\mathbb{E}[L|L > 0]$ to θ (left) and to the probability of loss ε (right).

1, as predicted by Proposition 3.6. This is illustrated in Figure 3a.

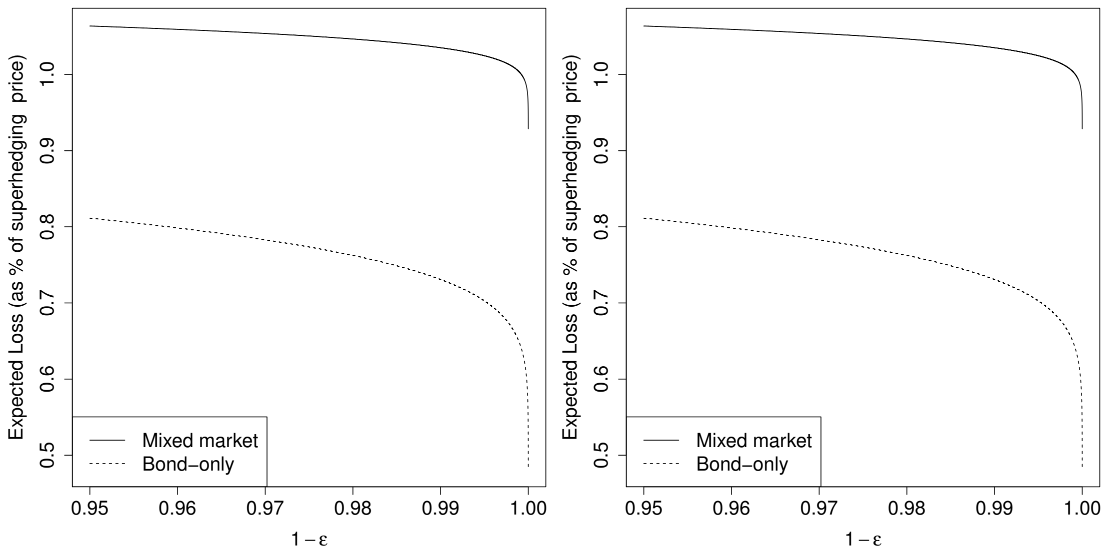
Since κ is a function of T, among others, changes in the maturity of the payoff will affect the behaviour of the expected loss. Figure 3b shows the conditional expected loss for $T \in [1, 40]$ when $\varepsilon = 0.01$. For lower values of T, the expected loss is lower in the bond-only market, but the opposite becomes true for large enough T.



(a) $T = 20$ (b) $\varepsilon = 0.01$

Figure 3: Sensitivity of $\mathbb{E}[L|L > 0]$ to the probability of loss ε (left) and to the maturity of the payoff T (right) when $\theta_r = 0.01$.

For the final part of this section, we return to the original parameter set presented in Table 1. We consider different crediting rates by changing g and n. The conditional expected loss for these two new payoffs is presented in Figure 4. On the left-hand side, we consider the expected loss resulting from a quantile hedging strategy for a zero-coupon bond with 20 years to maturity (that is, $n = g = 0$). On the right-hand side, the payoff considered is the accumulation of one unit of currency at the constant rate $g = 0.01$ (keeping $n = 0$). In both cases, as in Figure 2b, the mixed market expected loss is higher than its bond market counterpart. However, as a percentage of the superhedging price of the payoff, the expected loss is slightly higher when the crediting rate is no longer linked to the term structure ($n = 0$.) This can be explained by the lack of correlation between the payoff, which is now deterministic, and the bond market.



(a) $g = 0$, $n = 0$

(b) $g = 0.01$, $n = 0$

Figure 4: Sensitivity of $\mathbb{E}[L|L > 0]$ to the probability of loss ε, T=20.

5 Concluding remarks

In this paper, we applied the concept of quantile hedging introduced in (Föllmer and Leukert, 1999) to a particular type of pension payoff, which is linked to interest rates, taking into account survival of the plan member. Our goal was to assess the impact of the availability of equity on the cost of the quantile hedging strategy, and therefore to highlight the incentive resulting from a hedging criterion based solely on the probability of loss at maturity. The analysis of the performance of the quantile hedge was then used to illustrate the risk linked with the strategies yielding the lowest cost. Using simple models to represent bond-only and bond-and-equity markets allowed us to obtain explicit expressions for the cost of the optimal strategy. Thus, we identified a cost reduction when equity is available. We further gave expressions for the expected value of the unhedged loss, which we showed can increase when equity is used in the partial hedging strategy.

The criterion that we use to find optimal hedging strategies only relies on the probability of loss at maturity. Our numerical results show that this loss can be considerable, especially when equity can be added to the hedging portfolio, given that the interest rate risk premium is sufficiently high. However, including equity in the portfolio also reduces the cost of the hedge, which represents an important incentive to invest part of the hedging portfolio in risky assets. This result can be relevant in light of current or planned regulation based on the VaR risk measure, which also only takes into account the probability of loss. Further research should explore different efficient hedging criteria (for example, those presented in Barski (2016) and in Cong, Tan, and Weng (2014)) in the context of long-term pension claims to assess whether those modified criteria still encourage investment in equity by reducing the cost of the hedge. In particular, the size of the hedge should be taken into account.

The results obtained here depend on the market model we consider. Assuming specific dynamics for the short-rate and the risky asset allowed us to obtain explicit expressions that could be analysed. Completeness of the financial market is also a strong assumption that should be relaxed in future research. Adding incompleteness to the market through jumps in the risky asset price, for example, might increase the cost of the strategy in the mixed market.

Finally, our explicit results concern the initial cost of the hedging strategy. We do not present explicit expressions for the investment strategy. Because of the form of the payoff and of the completeness of the financial market, it is however possible to obtain expressions for the optimal quantile hedging strategy in each market.

A Proof of Proposition 2.1

A version of this result for a slightly different payoff H appears in Nakano (2011). The proof that we give here follows their proof closely.

Proof. First, set $\widetilde{x} = \mathbb{E}^*[Y]$ and let $\widetilde{\xi}$ be the perfect replicating strategy for Y (this strategy exists since $Y \in \mathcal{F}_T$). Then, since

$$\widetilde{x} + \int_0^T \widetilde{\xi}_s dX_s = Y,$$

and since $Y \geq H$, we have that $\widetilde{x} = \mathbb{E}^*[Y] \geq \Pi(H)$, by definition of $\Pi(\cdot)$.

Now consider some admissible strategy $\{x, \xi\}$ with

$$V_T^{x,\xi} = x + \int_0^T \xi_s dX_s \geq H.$$

Then, for any $\delta \in \mathcal{D}$, we have

$$\mathbb{E}[Z_T Z_T^\delta H] \leq \mathbb{E}[Z_T Z_T^\delta V_T^{x,\xi}] \leq x, \tag{A.1}$$

where the second inequality comes from the supermartingale property of $\{Z_t Z_t^\delta V_t^{x,\xi}\}$. By taking the infimum over x and by expressing the left-hand side of (A.1) as $\mathbb{E}[Z_T Z_T^\delta Y \mathbb{1}_{\{\tau > T\}}]$, we can write

$$\sup_{\delta \in \mathcal{D}} \mathbb{E}[Z_T Z_T^\delta Y \mathbb{1}_{\{\tau > T\}}] \leq \Pi(H).$$

But for any constant $\delta > -1$, we have

$$\begin{aligned}\mathbb{E}[Z_T Z_T^\delta Y \mathbb{1}_{\{\tau>T\}}] &= \mathbb{E}[Z_T Y(1+\delta \mathbb{1}_{\{\tau\le T\}})e^{-\delta\int_0^{\tau\wedge T}\lambda_s ds}\mathbb{1}_{\{\tau>T\}}] \\ &= \mathbb{E}[Z_T Y e^{-\delta\int_0^{T}\lambda_s ds}\mathbb{1}_{\{\tau>T\}}] \\ &= \mathbb{E}[Z_T Y \Lambda_T e^{-\delta\int_0^{\tau\wedge T}\lambda_s ds}] \\ &= \mathbb{E}[Z_T Y e^{-(\delta+1)\int_0^{\tau\wedge T}\lambda_s ds}].\end{aligned}$$

Then, $\mathbb{E}[Z_T Z_T^\delta Y \mathbb{1}_{\{\tau>T\}}] \to \mathbb{E}[Z_T Y]$ as $\delta \searrow -1$. It follows that $\mathbb{E}[Z_T Y] = \mathbb{E}^*[Y] \le \Pi(H)$. This ends the proof. □ □

B Proof of Proposition 2.3

To derive the distribution of $Z_T\zeta$, we first need to present the following lemmas. In Lemma B.1, we recall a result from stochastic calculus.

Lemma B.1. *Let $\{W(t)\}_{0\le t\le T}$ be a standard Brownian motion with $W(0)=0$, and fix $0\le t\le T$. Conditional on the value of $W(T)$, $W(t)$ follows a Gaussian distribution with*

$$\mathbb{E}[W(t)|W(T)\in dy] = \frac{t}{T}y, \tag{B.1}$$

and

$$Cov(W(t),W(s)|W(T)\in dy) = \min(t,s) - \frac{ts}{T}. \tag{B.2}$$

The result of Lemma B.1 can be obtained by first showing that the Brownian motion $\{W(t)\}_{0\le t\le T}$, conditioned on its value at T,[4] has the same distribution as a Brownian bridge. This can be proven by deriving the joint density of $W(t_1), W(t_2), \dots, W(t_n), W(T)$, with $0\le t_1\le t_2\le\dots\le t_n\le T$ and hence the density of $W(t_1), W(t_2), \dots, W(t_n)$ conditioned on the value of $W(T)$, which coincides with the density of a Brownian bridge reaching the same value at T. The interested reader is referred to Section 4.7.5 of Shreve (2004) for further details on the correspondence between the conditioned Brownian motion and the Brownian bridge. Once this correspondence is established, the result of Lemma B.1 follows from the distribution of the Brownian bridge, and can be found for example in Section IV.4 of Borodin and Salminen Borodin and Salminen (1996). It can also be obtained by integrating directly over the joint distribution of $W(s)$ and $W(t)$ conditioned on the value of $W(T)$. Below, we present a proof of (B.1), based on Section 4.7.5 of Shreve (2004). We omit the proof of (B.2), as it is lengthy and is not the main focus of the paper.

Proof. Let $p(t,x) = P(W(t)\in dx)$ denote the density of the Brownian motion at time $t\in[0,T]$, and recall that

$$p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}.$$

[4] It is important to note that here, we only condition on the value of the Brownian motion at time T, and not on its entire path from 0 to T, nor on the filtration induced by $\{W_t\}_{0\le t\le T}$.

Since the increments of the Brownian motion are independent and stationary, the joint density of $W(t)$ and $W(T)$ can be written as

$$\begin{aligned} P(W(t) \in dx, W(T) \in dy) &= p(t,x)p(T-t, y-x) \\ &= \frac{1}{\sqrt{2\pi t(T-t)}} e^{-\frac{1}{2}\left(\frac{x^2}{t}+\frac{(y-x)^2}{T-t}\right)}. \end{aligned}$$

Using the joint distribution of $W(t)$ and $W(T)$, the distribution of $W(t)$ conditional on the value of $W(T)$ can be obtained as follows.

$$\begin{aligned} P(W(t) \in dx | W(T) \in dy) &= \frac{P(W(t) \in dx, W(T) \in dy)}{P(W(T) \in dy)} \\ &= \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left\{-\frac{1}{2}\left(\frac{x^2}{t} + \frac{(y-x)^2}{T-t} - \frac{y^2}{T}\right)\right\} \\ &= \frac{1}{\sqrt{2\pi (t/T)(T-t)}} \exp\left\{-\frac{(x-y(t/T))^2}{2(t/T)(T-t)}\right\}. \end{aligned}$$

Therefore, conditional on $W(T) \in dy$, $W(t)$ has a normal distribution with mean $y\frac{t}{T}$, which is given in (B.1), and variance $\frac{t(T-t)}{T}$. □ □

Lemma B.2. *Let $\{W(t)\}_{0\le t\le T}$ be a standard Brownian motion with $W(0) = 0$ and denote $I(T) =: \int_0^T e^{at}W(t)dt$. Conditional on the value of $W(T)$, $I(T)$ follows a Gaussian distribution with*

$$\begin{aligned} \mathbb{E}\left[I(T)|W(T) \in dy\right] &=: y\mu_{I|W} \\ &= -\frac{y\ e^{aT}}{aT}(D(T)-T), \end{aligned}$$

and

$$\begin{aligned} Var\left(I(T)|W(T) \in dy\right) &=: s^2_{I|W} \\ &= \frac{e^{2aT}}{a^2}\left(D(T) - D^2(0,T)\left(\frac{a}{2}+\frac{1}{T}\right)\right). \end{aligned}$$

Proof. Using Lemma 2, we have that conditional on the value of $W(T)$, $e^{at}W(t)$ is Gaussian for any $t \in [0,T]$, and the integral $\int_0^T e^{at}W(t)dt$ is thus also Gaussian. Next, we obtain its expected value.

$$\begin{aligned} \mathbb{E}\left[\int_0^T e^{at}W(t)\ dt\middle| W(T) \in dy\right] &= \int_0^T e^{at}\mathbb{E}[W(t)|W(T) \in dy]\ dt \\ &= \int_0^T \frac{t\ e^{at}}{T} y\ dt \\ &= \frac{y}{T}\frac{e^{aT}(aT-1)+1}{a^2} \\ &= -e^{aT}\frac{y}{aT}(D(T)-T). \end{aligned}$$

Note that from (B.1) and (B.2), we have

$$\mathbb{E}[W(t)W(s)|W(T) \in dy] = \min(t,s) + st\left(\left(\frac{y}{T}\right)^2 - \frac{1}{T}\right). \tag{B.3}$$

To obtain the variance of $\int_0^T e^{at}W(t)dt$, it remains to obtain the expected value of its square.

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T e^{at}W(t)\ dt\right)^2 \middle| W(T) \in dy\right] \\
&\quad = \mathbb{E}\left[\left(\int_0^T e^{at}W(t)\ dt\right)\left(\int_0^T e^{as}W(s)\ ds\right) \middle| W(T) \in dy\right] \\
&\quad = \int_0^T \int_0^T e^{a(s+t)}\mathbb{E}[W(t)W(s)|W(T) \in dy] ds\ dt
\end{aligned}$$

Using (B.3) in the integral above and performing the relatively simple (but tedious) integration yields

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T e^{at}W(t)\ dt\right)^2 \middle| W(T) \in dy\right] \\
&\quad = \frac{e^{2aT}(2aT-3) + 4e^{aT} - 1}{2a^3} - \frac{(e^{aT}(aT-1)+1)^2}{a^4T} + \mathbb{E}\left[\int_0^T e^{aT}W(t)dt \middle| W(T) \in dy\right]^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&Var\left(\int_0^T e^{at}W(t)\ dt \middle| W(T) \in dy\right) \\
&\quad = \mathbb{E}\left[\left(\int_0^T e^{at}W(t)\ dt\right)^2 \middle| W(T) \in dy\right] - \mathbb{E}\left[\int_0^T e^{aT}W(t)dt \middle| W(T) \in dy\right]^2 \\
&\quad = \frac{e^{2aT}(2aT-3) + 4e^{aT} - 1}{2a^3} - \frac{(e^{aT}(aT-1)+1)^2}{a^4T}.
\end{aligned}$$

The desired results follows from re-arranging, and by using $D(T) = \frac{1-e^{aT}}{a}$, following the notation introduced for the zero-coupon bond price in Section 2.1.1. □ □

Lemma B.3. *Let $Y = \lambda_0 \int_0^T e^{at}W(t)dt + \lambda_1 W(T) + \lambda_2 \widetilde{W}(T)$, where λ_0, λ_1 and λ_2 are strictly positive constants, and $W(t)$ and $\widetilde{W}(t)$ are Brownian motions with $W(0) = \widetilde{W}(0) = 0$. Then Y follows a Normal distribution with mean 0 and variance*

$$\lambda_0^2(s_{I|W}^2 + \mu_{I|W}^2 T) + 2\lambda_0\mu_{I|W}\lambda_1 T + (\lambda_1^2 + \lambda_2^2)T,$$

where $s_{I|W}^2$ and $\mu_{I|W}^2$ are as defined in Lemma B.2.

Proof. This result can be shown by calculating the characteristic function of Y. Conditioning on the value of $W(T)$ and using independence of $W(t)$ and $\widetilde{W}(t)$, as well as the result given in Lemma B.2, will yield the desired result. □ □

We will now prove Proposition 2.3 using Lemmas B.2 and B.3.

Recall that ζ denotes the discounted purely financial payoff at maturity T. From (2.10) and (2.8), we have

$$\begin{aligned} Z_T\zeta &= \exp\left\{\boldsymbol{\theta}'\boldsymbol{W}(T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta} + \int_0^T \left(r^c(t;n) - r(t)\right)\, dt\right\} \\ &= \exp\left\{\eta(n,T) - \frac{T}{2}\boldsymbol{\theta}'\boldsymbol{\theta} + \widetilde{D}(n)\sigma_r e^{-aT}\int_0^T e^{at}W_r(t)dt + \boldsymbol{\theta}'\boldsymbol{W}(T)\right\}, \end{aligned} \tag{B.4}$$

where

$$\eta(n,T) = \begin{cases} gT - \frac{\gamma(n)T}{n} + \widetilde{D}(n)\left((r_0-b)\frac{1-e^{-aT}}{a} + bT\right), & n > 0, \\ gT - \left((r_0-b)\frac{1-e^{-aT}}{a} + bT\right), & n = 0, \end{cases} \tag{B.5}$$

and

$$\widetilde{D}(n) = \begin{cases} \frac{D(n)-n)}{n}, & n > 0 \\ -1, & n = 0. \end{cases}$$

To obtain (B.4), we solve (2.1) (see for example Section 3.2 of Brigo and Mercurio (2007)) and get

$$r(t) = e^{-at}\left(r_0 + b(e^{at}-1) - \int_0^t \sigma_r e^{au} dW_r(u)\right).$$

Integrating and using Itô's lemma gives

$$\int_0^T r(t)dt = (r_0-b)\frac{1-e^{-aT}}{a} + bT - \sigma_r e^{-aT}\int_0^T e^{at}W_r(t)dt,$$

which we use along with the definition of $y_n(t)$ to get (B.4) and (B.5).

The result follows from using Lemma B.3 with $\lambda_0 = \widetilde{D}(n)\sigma_r e^{-aT}$ and $(\lambda_1,\lambda_2)' = \boldsymbol{\theta}$.

C Proof of Proposition 3.4

In order to prove Proposition 3.4, we first need the two following lemmas.

Lemma C.1. *Let $\{W(t)\}_{t\geq 0}$ and $\{\widetilde{W}(t)\}_{t\geq 0}$ be independent Brownian motions, and let*

$$I(T) = \int_0^T e^{at}W(t)dt.$$

Then, $(\nu I(T), \nu I(T) + \theta_1 W(T) + \theta_2\widetilde{W}(T))$ has a bivariate Normal distribution with mean $(0,0)$ and covariance matrix

$$\begin{bmatrix} \nu^2 s_I^2 & \nu^2 s_I^2 + \nu\mu_{I|W}\theta_1 T \\ \nu^2 s_I^2 + \nu\mu_{I|W}\theta_1 T & \nu^2 s_I^2 + 2\nu\mu_{I|W}\theta_1 T + (\theta_1^2+\theta_2^2)T \end{bmatrix},$$

where $\mu_{I|W}$ and $s^2_{I|W}$ are as defined in Lemma B.2, and where $s_I^2 = (s^2_{I|W} + \mu^2_{I|W}T)$.

Proof. The joint distribution of $(\nu I(T), \nu I(T) + \theta_1 W(T) + \theta_2 \widetilde{W}(T))$ is obtained by calculating the moment generating function (MGF):

$$
\begin{aligned}
&\mathbb{E}\left[e^{\kappa_1 \nu I(T) + \kappa_2(\nu I(T) + \theta_1 W(T) + \theta_2 \widetilde{W}(T))}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{(\kappa_1+\kappa_2)\nu I(T) + \kappa_2(\theta_1 W(T) + \theta_2 \widetilde{W}(T))} | W(T), \widetilde{W}(T)\right]\right] \\
&= e^{\frac{1}{2}(\kappa_1+\kappa_2)^2 \nu^2 s^2_{I|W}} \mathbb{E}\left[e^{((\kappa_1+\kappa_2)\nu\mu_{I|W} + \kappa_2\theta_1) W(T)}\right] \mathbb{E}\left[e^{\kappa_2\theta_2 \widetilde{W}(T)}\right] \\
&= \exp\left\{\frac{1}{2}(\kappa_1+\kappa_2)^2 \nu^2 s^2_{I|W} + \frac{1}{2}((\kappa_1+\kappa_2)\nu\mu_{I|W} + \kappa_2\theta_1)^2 T + \frac{1}{2}\kappa_2^2\theta_2^2 T\right\} \\
&= \exp\left\{\frac{1}{2}\left(\kappa_1^2\nu^2 s_I^2 + 2\kappa_1\kappa_2(\nu^2 s_I^2 + \theta_1\nu\mu_{I|W} T) + \kappa_2^2(\nu^2 s_I^2 + 2\nu\mu_{I|W}\theta_1 T + (\theta_1^2 + \theta_2^2)T)\right)\right\}
\end{aligned}
$$

The desired result is obtained by comparing the expression obtained to the MGF of a bivariate normal distribution. □ □

Lemma C.2. *Let (X, Y) be bivariate Normal random variables with parameters $\mu = (0, 0)$ and*

$$
\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}.
$$

Let c_1, $c_2 \in \mathbb{R}$ be real constants. Then we have,

$$
\mathbb{E}\left[e^{c_1 X} \mathbb{1}_{\{Y > c_2\}}\right] = e^{\frac{1}{2}c_1^2\sigma_X^2} \Phi\left(\frac{c_1\rho\sigma_X\sigma_Y - c_2}{\sigma_Y}\right).
$$

Proof. The density of (X, Y) is given by

$$
f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right\}.
$$

This result follows by calculating the following integral:

$$
\mathbb{E}\left[e^{c_1 X} \mathbb{1}_{\{Y > c_2\}}\right] = \int_{c_2}^{\infty}\int_{-\infty}^{\infty} e^{c_1 x} f_{X,Y}(x, y) dx\ dy.
$$

□. □

A more general version of Lemma C.2 is presented in Theorem 4.1 of Melnikov and Romanyuk (2008); see also Lemma 4.1 of Melnikov (2011).

We can now prove Proposition 3.4. By Proposition 2.1, the superhedging portfolio for $\widetilde{H}_T = \mathbb{1}_{A_2(k^*)}\zeta(1 - N_T)$ is the replicating portfolio for $\mathbb{1}_{A_2(k^*)}\zeta$. Therefore, we have

$$
L = H_T - \widetilde{V}_T = \begin{cases} \zeta(1 - \mathbb{1}_{A_2(k^*)}), & \tau > T \\ -\zeta, & \tau \le T. \end{cases}
$$

Then, the set on which L is strictly positive is exactly $(A_2(k^*))^c \cap \{\tau > T\}$, and it follows that

$$
\mathbb{E}[L | L > 0] = \frac{E[\zeta(1 - \mathbb{1}_{A_2(k^*)})\mathbb{1}_{\{\tau > T\}}]}{P(A_2^c(k^*) \cap \{\tau > T\})} \tag{C.1}
$$

By Assumption 1, the survival process is independent of the financial market, and we have

$$P((A_2(k^*))^c \cap \{\tau > T\}) = (1 - P(A_2(k^*)))P(\tau > T) = \varepsilon.$$

The last equality results from the definition of k^* and the distribution of $Z_T\zeta$ given by (3.2) and Proposition 2.3, respectively.
Therefore, we can re-write (C.1) as

$$\begin{aligned}\varepsilon^{-1}E[\zeta(1-\mathbb{1}_{A_2(k^*)})\mathbb{1}_{\{\tau>T\}}] &= \varepsilon^{-1}E[\zeta(1-\mathbb{1}_{A_2(k^*)})\mathbb{1}_{\{\tau>T\}}|\tau > T]P(\tau > T)\\ &= \varepsilon^{-1}\Lambda_T E[\zeta(1-\mathbb{1}_{A_2(k^*)})\mathbb{1}_{\{\tau>T\}}]\end{aligned}$$

To obtain the expression presented in Proposition 3.4, it suffices to write H_T as a function of $I_r(T) = \int_0^T e^{at}W_r(t)dt$, and $\frac{dP^*}{dP}H_T$ in terms of $\nu I_r(T) + \theta_1 W_r(T) + \theta_2 W(T)$. The result of the proposition is obtained using the distribution presented in Lemma C.1, and the expectation given in Lemma C.2.

D Other useful results

Lemma D.1. *Let $\Phi(\cdot)$ denote the distribution of a standard Normal random variable, and let $z_{1-\varepsilon}$ denote the $1-\varepsilon$ quantile, so that $\Phi^{-1}(1-\varepsilon) = z_{1-\varepsilon}$. Then,*

$$\lim_{\varepsilon\to 0}\frac{\Phi(a - z_{1-\varepsilon})}{\varepsilon} = \begin{cases}\infty & a > 0,\\ 1 & a = 0,\\ 0 & a < 0.\end{cases}$$

Proof. First note that as ε goes to 0, $z_{1-\varepsilon}$ goes to infinity, so that $\lim_{\varepsilon\to0}\Phi(a - z_{1-\varepsilon}) = 0$. Using l'Hôpital's rule, we have

$$\lim_{\varepsilon\to 0}\frac{\Phi(a - z_{1-\varepsilon})}{\varepsilon} = \lim_{\varepsilon\to 0}\phi(a - \Phi^{-1}(1-\varepsilon))[\Phi^{-1}]'(1-\varepsilon),$$

where $\phi(\cdot) = \Phi'(\cdot)$, and $[\Phi^{-1}]'(x) = \frac{1}{\phi(\Phi^{-1}(x))}$. Therefore,

$$[\Phi^{-1}]'(1-\varepsilon) = \frac{1}{\phi(z_{1-\varepsilon})},$$

and it follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\Phi(a - z_{1-\varepsilon})}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\phi(a - z_{1-\varepsilon})}{\phi(z_{1-\varepsilon})}\\ &= \lim_{\varepsilon\to 0} e^{-\frac{1}{2}((a-z_{1-\varepsilon})^2 - z_{1-\varepsilon}^2)}\\ &= \lim_{\varepsilon\to 0} e^{-\frac{1}{2}a^2 + az_{1-\varepsilon}}\\ &= \lim_{x\to\infty} e^{-\frac{1}{2}a^2 + ax}.\end{aligned}$$

The result follows from considering the three cases $a < 0$, $a = 0$ and $a > 0$. □ □

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