Variable Annuities with VIX-linked Fee Structure under a Heston-type Stochastic Volatility Model

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Abstract

The Chicago Board of Options Exchange (CBOE) advocates linking variable annuity (VA) fees to its trademark VIX index in a white paper (CBOE, 2013a,b). It claims that the VIX-linked fee structure has several advantages over the traditional fixed percentage fee structure. However, the evidence presented in the white paper is largely based on non-parametric extrapolation of historical data on market prices. Our work lays out a theoretical basis with a parametric model to analyze the impact of the VIX-linked fee structure and to verify some claims from the CBOE white paper. In a Heston-type stochastic volatility setting, we jointly model the dynamics of an equity index (underlying the value of VA policyholders' accounts) and the VIX index. In this framework, we price a guaranteed minimum maturity benefit (GMMB) with VIX-linked fees. Through numerical examples, we show that the VIX-linked fee reduces the sensitivity of the insurer's liability to market volatility, when compared to a VA with the traditional fixed fee rate.

Key-words: Variable annuity, VIX index, dynamic fee, segregated funds, stochastic volatility, Heston model.

1 Introduction

Variable annuities (VAs) and many other equity-linked products in life insurance offer individual policyholders participation in the financial market while providing some protection

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against poor investment performance. While the industry used to perceive the guaranteed benefits as small add-ons with very little cost, equity-linked insurance products proved to carry significant long-term financial risks during the 2007-2008 financial crisis and the global recession that ensued. The crisis and its consequences on insurers highlighted the need for sound risk management strategies.

With regulation moving towards a "mark-to-market" valuation of liabilities, future costs and reserves for VAs are increasingly unpredictable, especially in times of financial turmoil. The financial guarantees embedded in variable annuities are typically financed through charges paid directly from policyholders' accounts. These fees are usually set as a fixed percentage of account values and deducted regularly until maturity. This fee structure often creates a misalignment between insurers' income streams and the market value of liabilities, which can in turn increase the cost of hedging and reduce the effectiveness of risk-management strategies.

While the long-term guarantees offered with VAs can be hedged using long-dated options, this strategy may be expensive because of the lack of suppliers of long-term derivatives in the financial market. Another way to hedge these long-term guarantees is to set up a hedging portfolio by rolling over short-dated options, which are more liquid and better suited to protect the insurer against short-term changes in implied volatility. However, such a hedge can become expensive in times of financial turmoil, since the cost of the new options can be high and unpredictable. For these reasons, the Chicago Board of Options Exchange (CBOE) has suggested, in a white paper (see CBOE 2013a; 2013b), to make the VA charge depend on the CBOE's trademark Volatility Index (VIX), which is a proxy for the 30-day forecast of market implied volatility of the S&P500 index option prices. The VIX is intended to measure the market's expectation of near-term stock market volatility and is often referred to as the "fear index" or the "gauge index". In the insurance industry, SunAmerica of AIG [a] has started to offer variable annuities with a fee rate linked to the VIX index.

The VIX index is typically negatively correlated with the underlying index price because of the well-documented *leverage effect*, i.e. the observed tendency of an asset's volatility to be negatively correlated with the asset's returns, especially when the index price drops. This phenomenon is well-documented in the empirical finance literature; see for example Section 7.3 of Rebonato (2005). Since the financial guarantees embedded in VAs are often similar to put options, their value increases when the underlying price drops and when the volatility rises. In other words, VA liabilities and their hedging costs are positively correlated with the VIX index. As mentioned by the CBOE (2013a), a VIX-linked fee structure would help re-align the value of the VA guarantees and the fees paid by the policyholder.

By re-aligning the value of the guarantee and the level of fees paid, a VIX-linked fee structure may also help manage lapsation risk. In general, a policyholder is more likely

[a]See slide 8 of http://www.cboermcus.com/uploads/5/2/6/5/52653589/52748_grissom-pres.pdf and the news coverage at http://retirementincomejournal.com/issue/february-3-2010/article/sunamerica-links-va-rider-fees-to-volatility-index and http://www.annuitydigest.com/reviews/intelligent-combination-income-plus-income-builder-and-vix

to surrender the VA contract when the perceived value of the guarantee is lower than the perceived expected future fees. With a fixed fee structure, more policies lapse when the market is stable, as policyholders perceive themselves as overpaying. In contrast, policyholders are less likely to surrender their policy under volatile market conditions as they feel insecure about their investments. This is similar to the phenomenon of *adverse selection* for traditional life insurance. With the VIX-linked fee structure, there is less incentive for policyholders to behave in such a way. In times of financial turmoil, high VIX index values lead to increased fees, reducing the incentive for the policyholder to remain in the contract. This reduces the adverse selection effect. This consequence of the VIX-linked fee structure is not explored further in the present paper, but could be investigated in future work.

Most of the academic literatures on VAs employ models based on the current market practice of a fixed percentage fee. However, dynamic fee structures that reduce the impact of policyholder behavior, another important risk factor for insurers, has recently been introduced in the literature. In that mindset, Bernard, Hardy, and MacKay (2014) consider a fee paid at a constant rate only when the VA account value is below a certain level under the Black-Scholes model. This work is extended to more general exponential Lévy market models in Delong (2014). MacKay, Augustyniak, Bernard, and Hardy (2015) show that this type of fees can be used to reduce an insurer's exposure to policyholder's behavior, which is an important risk factor. Bernard and MacKay (2015) study the impact on the surrender incentive of a fee set as a constant amount (as opposed to a constant rate) in the Black-Scholes model. By modifying the fee structure to reduce the impact of another major risk factor, market volatility in this case, our work is line with this growing body of literature. This paper distinguishes itself from prior work in several ways. First, our work analyzes the VIX-linked fee structure designed to mitigate volatility risk, while past work focused on policyholder behavior. Second, this paper presents a model for linking VA fees to a latent volatility factor, which requires a two-dimensional model for equity index and stochastic volatility, while previous work considered a fee structure dependent on policyholders' VA accounts, which is often based on a one-dimensional model.

To the authors' best knowledge, this is the first paper to study a VIX-linked fee structure in the academic literature.[b] The CBOE published a short non-parametric analysis of the suggested VIX-linked fee structure using historical index prices, which is an approach fundamentally different from the parametric modeling in this paper. Furthermore, the variable fee formula was set in an *ad hoc* manner in the CBOE report. Implementing such a fee structure in the absence of truly analytic models and without sound quantitative tools could potentially expose the variable annuity industry to significant systemic risk.

The *objective* of this paper is to fill this gap in the current literature by assessing the VIX-linked fee structure under a parametric stochastic volatility model that incorporates

[b]A working paper, Bernard, Kolkiewicz, and Tang (2016), has been brought to our attention. The authors propose a design that aligns the fee of a VA to the traded VIX index. In an empirical analysis, they fit the joint distribution of the VIX and the S&P500 index using a Gaussian copula, and carry out a parametric study based on the Black-Scholes model. Our present work takes into account the effects of stochastic volatility, and their model can be seen as a special case of ours.

practical features such as the leverage effect and the stochastic evolution of volatility. We develop a model example based on a guaranteed minimum maturity benefit (GMMB), in order to assess the efficacy of the new fee structure in reducing the insurer's exposure to market volatility. Many of the results in this paper can be extended to more complex guaranteed benefits in future research.

Our contribution to the literature is three-fold.

1. We provide a theoretical framework to quantify the effect of the VIX-linked fee structure by expressing the VA rider fee as a function of the parameters of a stochastic volatility model. We are thus able to write the dynamics of the underlying VA account only in terms of the market model parameters.

2. The characteristic function of the logarithm of the resulting account value is derived and hence used to obtain an expression for the price of a GMMB with VIX-linked fee. The underlying two dimensional model, which is an extension of the Heston model, has not been studied in previous literature.

3. The resulting analytic formulas provide an efficient tool to develop a numerical example. In contrast with the traditional fixed fee, we show that the VIX-linked fee reduces the sensitivity of the insurer's liability to market volatility and increases the robustness of future liabilities to changes in the long-term mean of the volatility.

The paper is organized as follows. Section 2 lays out a mathematical model that jointly models the S&P500 index and the VIX index, and introduces the VIX-linked fee structure. In Section 3, we derive the characteristic function for the log-value of a VA account, as well as expressions for the price and Greeks of a GMMB. Numerical examples are presented in Section 4 and illustrate the effect of the VIX-linked fee structure on the insurer's exposure to market volatility. Section 5 concludes this paper with a summary of the main findings as well as directions for future research.

2 Market model and fee structure

2.1 Market model and notation

In this paper, we consider a VA contract with a single investment account (which we often refer to as the "VA account") tracking the value of an equity index or an equity fund. We use the following notations:

- $\{S_t\}_{t\geqslant 0}$ – the underlying equity index;
- $\{F_t\}_{t\geqslant 0}$ – the VA account;
- $\{\mathrm{VIX}_t\}_{t\geqslant 0}$ – the value of the VIX index;
- $\{c_t^{tot}\}_{t\geqslant 0}$ – the total fee rate.

The total fee rate will be explained in details in Section 2.2 below. Throughout this paper, we assume that the equity index is the S&P500, since the VIX index is closely related to the value of this index. This is also the assumption used by the CBOE in their white papers (CBOE 2013a; 2013b). To assess and quantify the impact of a VIX-linked fee structure, we choose the market model for $\{S_t\}_{t\geqslant 0}$ with two criteria in mind. First, the model should include stochastic volatility in order to incorporate the dynamics of the VIX index in a consistent manner. Thus the VIX-linked fee will also be driven by stochastic dynamics, allowing for a more complete analysis of the effect of the fee structure. Second, among a large choice of stochastic volatility models, we choose one that is mathematically tractable in order to develop explicit analytical formulas for pricing and risk management purposes. It is widely acknowledged in the variable annuity industry that pricing and financial reporting based on Monte Carlo simulations can be extremely time-consuming and may require several layers of nested simulations. Closed-form expressions, which would only be available in parametric models, can be useful to speed up simulations and reduce computational burden, as they can replace pricing or computation of risk measures in inner loops of nested simulations. A hybrid use of a closed-form expression and Monte Carlo simulations is presented with a numerical example in Section 4.

An ideal candidate meeting both of the above-mentioned criteria is the Heston stochastic volatility model (Heston (1993)). We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t, t\geqslant 0\}$, where $\mathbb{P}$ is the objective (real-world) measure. In the Heston model, the index S_t has the following dynamics

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu\ dt + \sqrt{V_t}d\widetilde{W}_t^{(1)},\\ dV_t &= \kappa^*(\bar{V}^* - V_t)dt + \sigma\sqrt{V_t}d\widetilde{W}_t^{(2)},\end{aligned} \tag{1}$$

where μ is the drift term representing physical return, $\kappa^*>0$ is the mean-reversion rate at which the variance process tends to move towards its long-term mean $\bar{V}^*>0$, and $\sigma>0$ is the "volatility of volatility" parameter. We assume that the $\mathbb{P}$-Brownian motions $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ have constant correlation $-1\leqslant\rho\leqslant 1$, which means that the cross variation satisfies $[\widetilde{W}_t^{(1)}, \widetilde{W}_t^{(2)}] = \rho t$. Due to the leverage effect, the correlation ρ is usually negative for stock or equity index prices. In other words, high market volatility is generally associated with low returns.

The process V_t is the latent stochastic variance process. It follows the well-known Cox-Ingersoll-Ross (CIR) process, which was first introduced in Cox, Ingersoll, and Ross (1985) to model the evolution of interest rates. It has the desirable property that it never breaches its lower boundary at zero if the Feller condition $2\kappa^*\bar{V}^*\geqslant\sigma^2$ is satisfied. This condition is a consequence of the Feller's test of explosions (Theorem 5.5.29, page 348 of Karatzas and Shreve (1991)). The CIR process also exhibits mean-reversion, which is consistent with the observation of variance in the empirical finance literature.

In the Heston model, the presence of stochastic volatility leads to market incompleteness, so there exists an infinite number of equivalent martingale measures. The risk-neutral measure used for pricing purposes is obtained by specifying a so-called "market price of volatility risk" $\Lambda(S, V, t)$, which is assumed to be proportional to the volatility. In other words, $\Lambda(S,V,t) = \lambda\sqrt{V}$ for some constant λ. For more details on this change of measure, see Heston (1993) and Gatheral (2006). With such a parameter, the dynamics of the underlying index and of the instantaneous volatility have the same form under the real-world measure and the risk-neutral one, albeit with different parameters.

Then under the resulting $\mathbb{Q}$-measure, the underlying index value under the Heston model has the following dynamics

$$\begin{aligned}\frac{dS_t}{S_t} &= r\ dt + \sqrt{V_t}dW_t^{(1)},\\ dV_t &= \kappa(\bar{V} - V_t)dt + \sigma\sqrt{V_t}dW_t^{(2)},\end{aligned} \tag{2}$$

where r is the risk-free interest rate, $\kappa = \kappa^* + \lambda$, $\bar{V} = \kappa^*\bar{V}^*/(\kappa^* + \lambda)$, and $W_t^{(1)}$ and $W_t^{(2)}$ are $\mathbb{Q}$-Brownian motions with cross variation $[W_t^{(1)}, W_t^{(2)}] = \rho t$. Note that under the $\mathbb{Q}$-measure, σ reflects the "significance" of the volatility skew effects, and higher values of σ will lead to more prominent skew effects in the implied volatility surface. For further details on the Heston model, interested readers can refer to Gatheral (2006) and the references therein.

2.2 VA fee schedule

We assume that the fee rate paid from the VA account is composed of two parts. The first part, called *investment management fee*, is used to compensate the managers of the underlying investment funds (such as a mutual fund or an ETF) for their services. The second part, called *rider fee* or *rider charge*, goes to the insurer in order to cover the cost of additional investment guarantees (or riders). So far, our setting is similar to the one used, for example, in Chen, Vetzal, and Forsyth (2008). However, while they keep both parts of the fee rate fixed, we assume that the rider fee varies with the market volatility.

We denote the rate of total fees paid out of the fund by c_t^{tot}. Therefore, we have

$$c_t^{tot} = c^{inv} + c_t, \tag{3}$$

where c^{inv} and c_t are the rates of the investment management fee and of the rider charge, respectively. Throughout the paper, we assume that $c^{inv} \geqslant 0$ is constant and determined at $t = 0$.

2.2.1 VIX-linked rider charge

The CBOE Volatility Index (VIX) was introduced by the CBOE in 2003 with the purpose of measuring the market's expectation of the 30-day volatility implied from at-the-money S&P500 Index (SPX) option prices. As such, it is a proxy for the volatility of the equity market. The VIX is calculated as the weighted average of SPX call and put prices over a wide range of strikes. For the exact formula used by the CBOE for the calculation of the VIX, see for example equation (1) on page 4 of the white paper CBOE (2013a).

The square of the VIX, denoted by VIX^2, can be understood with mathematical simplifications as the risk-neutral expectation of the average of the integrated variance over the next 30 days[c]. Thus, in the Heston model presented above, we express the VIX squared at time $t \geqslant 0$ as

$$\text{VIX}_t^2 = \mathbb{E}_t\left[\frac{1}{\tau}\int_t^{t+\tau} V_s ds\right], \tag{4}$$

where $\tau = 30/365$. For brevity of notation, we write $\mathbb{E}_t[\cdot]$ for $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$. Proposition 5.1 of Zhu and Zhang (2007) allows us to write this expectation as a function of the current instantaneous variance V_t

$$\mathbb{E}_t\left[\frac{1}{\tau}\int_t^{t+\tau} V_s ds\right] = A + BV_t, \tag{5}$$

where

$$A = \frac{\bar{V}(\kappa\tau - 1 + e^{-\kappa\tau})}{\kappa\tau}, \qquad B = \frac{1 - e^{-\kappa\tau}}{\kappa\tau}. \tag{6}$$

Thus, we can re-write the square of the VIX index at time t as a linear function of V_t

$$\text{VIX}_t^2 = A + BV_t. \tag{7}$$

The purpose of the VIX-linked fee rate is to increase the fee income when the value of VA guarantee rises, in order to compensate for the heightened level of financial risk. Such values usually occur when the volatility of the equity index is high. To achieve a positive correlation between the fee and the liability, we want the variable fee to be an increasing function of the VIX index.

While there are other ways to link the rider fee rate to the VIX index, we consider a linear function of the squared VIX for its mathematical tractability. In particular, we assume that the VIX-linked rider fee c_t is determined by

$$c_t := \bar{c} + m \cdot \text{VIX}_t^2, \tag{8}$$

where $\bar{c} > 0$ is the "base fee rate" and $m > 0$ is the "multiplier". Intuitively speaking, as the VIX is an index of volatility, its square is a measure of the variance of market stock prices. This set of two parameters, $(\bar{c}, m)$, provide flexibility in setting the fee structure. As we will show in Section 4, a fair fee structure (see Definition 3.1) associates a higher base fee rate $\bar{c}$ with a lower multiplier m. Therefore, the level of the base fee rate is related to the sensitivity of the rider fee income to the VIX index. By setting the fee rate $\bar{c}$, an insurer can choose the level of dependence of the fee income on the VIX index.

It follows immediately from (8) that the VIX-linked fee rate is always non-negative, regardless of the fluctuation in the VIX index. Using (3), (7) and (8), one can rewrite

$$c_t^{tot} = \beta + \alpha(V_t - \bar{V}), \tag{9}$$

[c]See equation (5) on page 524 of Zhang and Zhu (2006).

where $\alpha := m(1 - e^{-\kappa\tau})/(\kappa\tau) > 0$ and $\beta := c^{inv} + \bar{c} + m \cdot \bar{V}$.

Note that $\bar{V}$ is the long-term mean of variance for the mean-reverting variance process $\{V_t, t \geqslant 0\}$. Therefore, the constant β in (9) can be viewed as the "long-term average fee rate" and α can be interpreted as the "volatility risk premium" rate, at which the rider fee changes with the deviation of variance from its long-term mean. The positivity of α implies that policyholders will be charged more to compensate for the additional risk undertaken by insurers due to high volatility in the market; a higher instantaneous volatility leads to a higher fee rate. This is clearly in agreement with the intended purpose of the VIX-linked fee structure in (8), which aims for a better alignment of fee income and liability. A numerical illustration of this relationship can be found in Section 4.4.

2.3 Dynamics of the fund with VIX-linked fees

As the fees are paid out of the fund at the rate c_t^{tot} defined in (3) and (8), the total (accumulated) amount of fees paid up to time t, denoted by C_t^{tot}, is given by

$$dC_t^{tot} = c_t^{tot} F_t \, dt. \tag{10}$$

Throughout the paper, we assume that the full initial premium is invested in the fund tracking the equity index, and that no further premiums are deposited later on. The instantaneous change in the VA account at any point in time is composed of two parts – the instantaneous return from the investment in the equity index, and the instantaneous deduction of fees.

$$\frac{dF_t}{F_t} = \frac{dS_t}{S_t} - \frac{dC_t^{tot}}{F_t} = \frac{dS_t}{S_t} - c_t^{tot} \, dt.$$

It follows immediately from (9) that the risk-neutral dynamics of the policyholder's account value is given by the following system of SDEs:

$$\begin{aligned} \frac{dF_t}{F_t} &= [r - \beta - \alpha(V_t - \bar{V})]dt + \sqrt{V_t} dW_t^{(1)}, \\ dV_t &= \kappa(\bar{V} - V_t)dt + \sigma\sqrt{V_t} dW_t^{(2)}. \end{aligned} \tag{11}$$

The VIX-linked fee model (11) is new to the literature[d] and is distinguished from the original Heston stochastic volatility model by the presence of a linear function of V_t in the drift of the account value process. It should be noted that the derivation of the characteristic function related to (11) does not follow immediately from its original Heston model counterpart (see Heston (1993)), and that it requires non-trivial manipulations. See Proposition 3.1 for more details.

The model presented in (11) includes simpler models as special cases. For these models, closed-form expressions for option prices are well documented in the literature and can be used as benchmarks for numerical testing.

[d] In our setting, it would be straightforward to add a constant dividend yield, if one desires, by making the appropriate change in the drift of the index price, and in the definition of β.

1. If we set $\sigma = \kappa = 0$, the fund value process becomes a geometric Brownian motion with drift parameter $r - \beta - \alpha(V_0 - \bar{V})$ and volatility parameter $\sqrt{V_0}$.

2. If we set $m = 0$, then it implies $\alpha = 0$. Thus, from (11), the resulting drift rate is $r-\beta$, which is consistent with the original Heston model assuming a constant dividend yield of β. Characteristic functions of the log stock price in the Heston model and related option pricing have been studied in many papers, such as Albrecher, Mayer, Schoutens, and Tistaert (2007), and Lord and Kahl (2010).

2.4 Dynamics of the fund under the real-world measure

The main goal of this paper is to assess the effect of the VIX-linked fee on the liability of the GMMB, which is calculated under the risk-neutral measure. However, for risk management purposes, we are also interested in the distribution of future liabilities under the real-world measure, since it represents the uncertainty with the future value of the insurer's financial obligations. Indeed, while the risk-neutral measure is an appropriate tool for pricing purposes, it is not representative of the actual evolution of the risk factors that affect insurers. For this reason, we also need to introduce the dynamics of the account F_t under the real-world measure $\mathbb{P}$.

In order to incorporate the VIX-linked fee in the real-world dynamics of the fund, we re-write the fee defined in (9) in terms of the real-world parameters κ^* and $\bar{V}^*$:

$$c_t^{tot} = \beta^* + \alpha^*(V_t - \bar{V}^*), \tag{12}$$

with

$$\begin{aligned}
\alpha^* &:= m\ h(\tau(\kappa^* + \lambda)), \\
\beta^* &:= c^{inv} + \bar{c} + m\bar{V}^* \left(\frac{\kappa^*}{\kappa^* + \lambda}(1 - h(\tau(\kappa^* + \lambda))) + h(\tau(\kappa^* + \lambda)) \right),
\end{aligned}$$

where $h(x) = (1 - e^{-x})/x$. This notation allows us to keep the same form for c_t under both the measures $\mathbb{Q}$ and $\mathbb{P}$. To obtain the real-world dynamics of the VA account, we use (1) and obtain

$$\frac{dF_t}{F_t} = (\mu - \beta^* - \alpha^*(V_t - \bar{V}^*))\ dt + \sqrt{V_t} d\widetilde{W}_t^{(1)}, \tag{13}$$

with V_t as given in (1).

3 Pricing and hedging the GMMB

In this section, we derive an expression for the value of a GMMB written on a VA contract with a VIX-linked fee structure. We also give results pertaining to the Greeks of the GMMB.

3.1 Pricing the GMMB

Consider a GMMB rider, which guarantees the policyholder the greater of a guaranteed amount G and the fund value F_T at maturity T of the VA policy, if she is still alive. Common types of guarantees include:

- A full refund of the initial premium: $G = F_0$;
- A refund of initial premium with "roll-up" at the rate δ: $G = F_0 e^{\delta T}$.

Since the policyholder's account is typically managed by a third-party investment manager, the insurer is only responsible for the difference between the guaranteed amount G and the account value F_T at maturity when the former exceeds the latter, usually due to poor market performance. Therefore, if the policyholder is still alive, the cost at maturity of such a guarantee, from the insurer's perspective, is equal to

$$(G - F_T)_+ := \max(G - F_T, 0). \tag{14}$$

Denote by τ_x the random variable representing the future lifetime of a policyholder aged x at inception of the VA contract, which is assumed to be independent of the underlying dynamics. Then, using the strong Markov property, it is easy to show that there exists a function $\Pi := \Pi(t, f, v)$ such that the no-arbitrage value of the guarantee at time t is given by

$$\Pi(T - t, F_t, V_t) = \mathbb{E}_t[e^{-r(T-t)}(G - F_T)_+ \mathbb{1}_{\{\tau_x > (T-t)\}}], \tag{15}$$

where $\mathbb{1}_A = 1$ on the set A and 0 elsewhere.

It is assumed that the insurer sells a large enough number of VA policies of similar sizes to a homogeneous population so that the idiosyncratic mortality risk is diversified. Even though the VA contracts are all linked to the same equity index, we can still apply an extended version of the strong law of large numbers, as discussed in full details in Feng and Shimizu (2016) and Feng (2014). Then, as in MacKay, Augustyniak, Bernard, and Hardy (2015), the value of the guarantee (15) can be written as[e]

$$\Pi(T - t, F_t, V_t) = {}_{T-t}p_x \mathbb{E}_t[e^{-r(T-t)}(G - F_T)_+], \tag{16}$$

where ${}_{T-t}p_x = \mathbb{Q}(\tau_x > T - t)$. In this paper, we focus on the volatility risk and its effect on the expectation in (16). Therefore, in the following we assume ${}_{T-t}p_x = 1$. This assumption could easily be relaxed and would only change our results by a constant.

It is clear that the computation of the cost of the guarantee in (16) requires the risk-neutral distribution of F_T. Thus, we consider the stochastic process $X_t := \ln(F_t/F_0)$, $0 \leqslant t \leqslant T$ and derive the conditional characteristic function $\varphi(u, t) := \mathbb{E}_t^Q[e^{iuX_T}]$. Whenever necessary, we write $\varphi(u, t; T, V_t)$ to emphasize its dependence on the parameters T and V_t.

[e] Here we further assume that market and biometric risks are independent, and that survival probabilities are the same under the real-world measure and the risk-neutral measure.

3.1.1 Characteristic function of log account value

The following is the main result of this paper, and provides the characteristic function of the log of the scaled account value $X_T := \ln(F_T/F_0)$ conditional on $\mathcal{F}_t$.

Proposition 3.1. *The conditional characteristic function of X_T is given explicitly by*

$$\varphi(u,t) = \exp\left\{ iuX_t + iu(r-\beta+\alpha\bar{V})(T-t) + \kappa\bar{V}(T-t)\frac{q}{\sigma^2} + \frac{2\kappa\bar{V}}{\sigma^2}\ln\frac{1-g}{1-ge^{-d(T-t)}} + \frac{V_t q}{\sigma^2}\frac{1-e^{-d(T-t)}}{1-ge^{-d(T-t)}} \right\}, \tag{17}$$

where $d = d(u) := \sqrt{(\kappa - i\rho\sigma u)^2 + \sigma^2((2\alpha+1)iu + u^2)}$, $q = q(u) := \kappa - d - i\rho\sigma u$, and $g = g(u) := q/(q+2d)$.

The proof of Proposition 3.1 can be found in Appendix A. For notational convenience, in the following we denote $\varphi(u) := \varphi(u,0)$.

Remark 3.1. *While there are other representations of the same characteristic function as seen in the proof of Proposition 3.1, we have chosen the formulation in* (17) *for which the fraction $G(u,t) := (1-g(u))/(1-g(u)e^{-d(u)(T-t)})$ does not move across the branch cut of the complex logarithmic function $(-\infty, 0]$. The advantage of such a formula over other equivalent forms is that it only utilizes the principal branch of the logarithmic function, which is built into most mathematical programming platforms, such as Maple and Mathematica.*

A similar technical problem and treatment appeared in the Heston model, as discussed in details in Albrecher, Mayer, Schoutens, and Tistaert (2007), and Lord and Kahl (2010). Note that if we take $\beta = \alpha = 0$ in (17)*, the formula reduces to the second formulation in equation (2) on page 84 of Albrecher, Mayer, Schoutens, and Tistaert (2007). This is intuitive since our model in* (11) *reduces to the original Heston model when $\beta = \alpha = 0$.*

Remark 3.2. *It is straightforward to generalize the above analysis in Proposition 3.1 to the case of deterministic time-dependent interest rates to account for the term structure of interest rates. i.e. we consider $r_s, s \geqslant 0$. The only modification we need to make is to replace $r(T-t)$ by $\int_t^T r_s ds$ wherever they appear. It is also possible to extend the analysis to account for stochastic interest rates, where we assume that r follows a stochastic process possibly correlated with the fund F and the volatility V. For simplicity, we do not present the required analysis, which can be carried out using ideas similar to the ones in van Haastrecht, Lord, Pelsser, and Schrager (2009).*

3.1.2 Risk-neutral valuation of the GMMB

Using the characteristic function of X_T, we can develop a Black-Scholes type formula for the time-t value of the GMMB. It follows from (14) and the strong Markov property that

$$\begin{aligned}\Pi(T-t, F_t, V_t) &= e^{-r(T-t)}\mathbb{E}_{(F_t,V_t)}[(G-F_T)_+] \\ &= e^{-r(T-t)}\left(G\Pi_1(T-t,F_t,V_t) - F_t\Pi_2(T-t,F_t,V_t)\right),\end{aligned} \tag{18}$$

where

$$\Pi_1(T-t,f,v) := \mathbb{Q}(X_T \leqslant \ln(G/f)|F_t = f, V_t = v),$$
$$\Pi_2(T-t,f,v) := \mathbb{E}_{(f,v)}[e^{X_T} I\big(X_T \leqslant \ln(G/f)\big)],$$

and here $\mathbb{E}_{(f,v)}[\cdot]$ is short-hand notation for $\mathbb{E}[\cdot|F_t = f, V_t = v]$. The two components $\Pi_1(t,f,v)$ and $\Pi_2(t,f,v)$ both rely on the characteristic function $\varphi(u,t)$.

Proposition 3.2. *Denote $k = \ln(G/f)$, then we have*

$$\Pi_1(T-t,f,v) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi(u,t)}{iu}\right] du, \tag{19}$$

$$\Pi_2(T-t,f,v) = \varphi(-i,t)\left[\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi(u-i,t)}{iu\varphi(-i,t)}\right] du\right]. \tag{20}$$

The proof of Proposition 3.2 can be found in Appendix B.

3.1.3 Fair fee structure

A fair fee structure for a VA guarantee is set such that, at $t = 0$, the risk-neutral value of the total income to the insurer is equal to that of the guarantee (the insurer's total outflow).

Definition 3.1. *A fair fee structure is a pair $(\bar{c}^*, m^*)$ that satisfies*

$$\mathbb{E}\left[\int_0^T e^{-ru} c_u F_u^{(\bar{c},m)} du\right] = \Pi^{(\bar{c},m)}(T, F_0, V_0), \tag{21}$$

where we add the superscript $(\bar{c}, m)$ to highlight the dependence of F_t and of the value of the guarantee on the fee structure.

Recall that the rate of total fees c_u^{tot} is shared between the investment manager and the insurer. Thus we only include the rider fees on the left-hand side of (21). Note that the fair fee structure is not unique, since the fee structure is described by two parameters, $\bar{c}$ and m. However, for fixed $\bar{c}$, there exists at most one fair m, and for a fixed m, there is at most one fair $\bar{c}$. Throughout this paper, we will use the asterisk superscript to indicate a fair fee couple $(\bar{c}^*, m^*)$ satisfying (21).

The result presented in the following proposition shows that the left-hand side of (21) can be written as a deterministic integral in one dimension.

Proposition 3.3. *The risk-neutral value of the rider fees conditional on $\mathcal{F}_t$, $0 \leqslant t \leqslant T$, can be expressed by*

$$\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} c_u F_u du\right] = F_t - e^{-r(T-t)}\varphi(-i,t;T) - c^{inv}\int_t^T e^{-r(u-t)}\varphi(-i,t;u)du. \tag{22}$$

The proof of Proposition 3.3 can be found in Appendix C.

Remark 3.3. *The presence of the investment management fee, c^{inv}, complicates the calculation of the fee income to the insurer. When $c^{inv} = 0$, the value of the fee income is linked to the current value of the fund and the discounted expectation of its final value (see Liu (2010) for more details). However, when $c^{inv} > 0$, this equivalence breaks down and the value of the fee income becomes path-dependent. In such a situation, the distribution of the whole path of the fund is required. In our setting, this results in the addition of the last term on the right-hand side of* (22)*. This involves the distribution of F_u, which is represented by its characteristic function evaluated at $-i$, at all times between t and T. It follows that with investment management fees, the calculation of the fee income requires multiple evaluations of the characteristic function, as well as additional numerical integration.*

3.2 Greeks of the GMMB

Variable annuity writers usually develop hedging programs to mitigate the risks embedded in VA riders. In particular, Greeks-based methods developed to hedge derivatives are often used to manage financial market risk, because many VA riders can be viewed as long-dated options. Due to the complexity of product designs, practitioners often use Monte Carlo simulations to estimate the Greeks, which can be extremely time-consuming, and somewhat inaccurate because of sampling errors and biases. In this section, we derive expressions for commonly used Greeks of the GMMB with a VIX-linked fee structure.

In practice, the delta of an option is defined as the rate of change of the option value with respect to changes in the underlying asset price, with everything else including time and volatility being fixed. When working with guarantees written on the underlying VA account, this definition of delta poses a challenge, because the no-arbitrage cost at time t of the GMMB depends on S_t only through F_t (see Proposition 3.2). Recall that the instantaneous change in F_t is attributable to (1) equity-linked financial returns $F_t(dS_t/S_t)$ due to changes in the underlying equity index and (2) the collection of fee income $c_t dt$ due to the passage of time, i.e. we have

$$\frac{dF_t}{F_t} = \frac{dS_t}{S_t} - c_t^{tot} dt.$$

Recall that in the original definition of delta, the time parameter is considered as a fixed parameter. To eliminate the time effect on the changes of F_t, we have to interpret changes in the fund value F due to changes in the equity index S as

$$dF/F = dS/S, \qquad \text{or equivalently} \qquad dF/dS = F/S.$$

With this motivation, we define the delta of the GMMB rider as

$$\Delta_t^M := \frac{F_t}{S_t}\frac{\partial \Pi}{\partial f}(T-t, F_t, V_t). \tag{23}$$

The vega of an option is the rate of change of its value with respect to changes in the instantaneous volatility V_t. It is of particular interest to our setting, because the VIX-linked fee structure aims at reducing the sensitivity of the liabilities to changes in the volatility. The vega of the GMMB guarantee is defined as

$$\mathcal{V}_t^M := \frac{\partial \Pi}{\partial v}(T-t, F_t, V_t). \tag{24}$$

The rho of an option is the rate of change of its value with respect to changes in the risk-free interest rate r. It is of particular importance since most variable annuity contracts are long-term contracts. The rho of the GMMB guarantee is defined as

$$\mathcal{R}_t^M := \frac{\partial \Pi}{\partial r}(T-t, F_t, V_t). \tag{25}$$

We can then use (18) and Proposition 3.2 to obtain expressions for the Greeks of the GMMB guarantee.

Corollary 3.1. *The delta of the GMMB guarantee is given by*

$$\Delta_t^M = \frac{F_t e^{-r(T-t)}}{S_t}\left[G\frac{\partial \Pi_1}{\partial f}(T-t, F_t, V_t) - \Pi_2(T-t, F_t, V_t) - F_t\frac{\partial \Pi_2}{\partial f}(T-t, F_t, V_t)\right],$$

where

$$\begin{aligned}
\frac{\partial \Pi_1(T-t, f, v)}{\partial f} &= \frac{1}{-\pi f}\int_0^\infty \Re\left[e^{-iuk}\varphi(u,t)\right]\,du, \\
\frac{\partial \Pi_2(T-t, f, v)}{\partial f} &= \frac{1}{-\pi f}\int_0^\infty \Re\left[e^{-iuk}\varphi(u-i,t)\right]\,du.
\end{aligned}$$

The vega is given by

$$\mathcal{V}_t^M = \frac{F_t e^{-r(T-t)}}{S_t}\left[G\frac{\partial \Pi_1}{\partial v}(T-t, F_t, V_t) - F_t\frac{\partial \Pi_2}{\partial v}(T-t, F_t, V_t)\right],$$

where

$$\begin{aligned}
\frac{\partial \Pi_1(T-t, f, v)}{\partial v} &= \frac{-1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi_v(u,t)}{iu}\right]du, \\
\frac{\partial \Pi_2(T-t, f, v)}{\partial v} &= \varphi_v(-i,t)\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi(u-i,t)}{iu\varphi(-i)}\right]du\right. \\
&\quad \left. + \varphi(-i,t)\left(-\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}(\varphi_v(u-i,t)\varphi(-i,t) - \varphi(u-i,t)\varphi_v(-i,t))}{(\varphi(-i,t))^2}\right]du\right)\right\},
\end{aligned}$$

and

$$\varphi_v(u,t) = \varphi(u,t)\frac{q(u)}{\sigma^2}\left(\frac{1-e^{-d(u)(T-t)}}{1-g(u)e^{-d(u)(T-t)}}\right). \tag{26}$$

The rho is given by

$$\mathcal{R}_t^M = -(T-t)\Pi(T-t,F_t,V_t) + e^{-r(T-t)}\left[G\frac{\partial \Pi_1}{\partial r}(T-t,F_t,V_t) - F_t\frac{\partial \Pi_2}{\partial r}(T-t,F_t,V_t)\right],$$

where

$$\begin{aligned}
\frac{\partial \Pi_1(T-t,f,v)}{\partial r} &= \frac{-1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi_r(u,t)}{iu}\right]du,\\
\frac{\partial \Pi_2(T-t,f,v)}{\partial r} &= \varphi_r(-i,t)\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\varphi(u-i)}{iu\varphi(-i,t)}\right]du\right.\\
&\quad\left. + \varphi(-i,t)\left(-\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}(\varphi_r(u-i,t)\varphi(-i,t) - \varphi(u-i,t)\varphi_r(-i,t))}{(\varphi(-i,t))^2}\right]du\right)\right\},
\end{aligned}$$

and $\varphi_r(u,t) = (iu(T-t))\varphi(u,t)$.

3.3 Greeks of the net liability of the GMMB

The previous section considers the sensitivity of the insurer's GMMB *gross liability*; it only takes into account its future payouts. However, one has to be reminded that, unlike exchange-traded options, the GMMB rider is compensated by a stream of fee income, which helps cover the cost of the gross liability. There are at least two problems with merely hedging the gross liability. Some discussion of gross and net liabilities can be found in Feng and Volkmer (2012).

1. The financial risk affects both the gross liability and the fee income. A hedging program developed only for the gross liability overlooks the uncertainty from the income side.

2. In most cases, small GMMB rider payouts can be covered by the accumulated fee income, leading to a profit for the insurer. However, a hedging program developed for the gross liability would completely eliminate even these small payouts. In that case, such an offset would be considered excessive. This indicates that a hedging program that does not take fee income into consideration is more costly than necessary.

For a prudent risk management strategy, the insurer should hedge its *net liability*, defined as the difference between the gross liability (the GMMB benefit) and the fee income coming from the rider part of the total fee:

$$\Pi^{\text{Net}}(T-t,F_t,V_t) := \Pi(T-t,F_t,V_t) - \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c_u F_u du\right]. \tag{27}$$

As explained in Section 3.1.3, the investment management fee c^{inv} increases the complexity of the computation of the fee income to the insurer.

Using Proposition 3.3, we can write the net liability as

$$\Pi^{\mathrm{Net}}(T-t,F_t,V_t)=\Pi(T-t,F_t,V_t)-F_t+e^{-r(T-t)}\varphi(-i,t;T)+c^{inv}\int_t^T e^{-r(u-t)}\varphi(-i,t;u)\ du. \tag{28}$$

Corollary 3.2. *The delta, vega and rho of the net liability, defined as*

$$\begin{aligned}\Delta_t^{Net}&:=\frac{F_t}{S_t}\frac{\partial \Pi^{Net}}{\partial f}(T-t,F_t,V_t),\\ \mathcal{V}_t^{Net}&:=\frac{\partial \Pi^{Net}}{\partial v}(T-t,F_t,V_t),\\ \mathcal{R}_t^{Net}&:=\frac{\partial \Pi^{Net}}{\partial r}(T-t,F_t,V_t)\end{aligned}$$

are given by

$$\begin{aligned}\Delta_t^{Net}&=\Delta_t^{M}-\frac{F_t}{S_t}+\frac{e^{-r(T-t)}}{S_t}\varphi(-i,t;T)+\frac{c^{inv}}{S_t}\int_t^T e^{-r(u-t)}\varphi(-i,t;u)\ du,\\ \mathcal{V}_t^{Net}&=\mathcal{V}_t^{M}+e^{-r(T-t)}\varphi_v(-i,t;T)+c^{inv}\int_t^T e^{-r(u-t)}\varphi_v(-i,t;u)\ du\\ \mathcal{R}_t^{Net}&=\mathcal{R}_t^{M},\end{aligned}$$

where $\varphi_v(\cdot)$ is defined in (26).

Proof. The results follow from the definition of $\Pi^{\mathrm{Net}}(T-t,F_t,V_t)$ in (28). Note that the second part of (28) does not depend on the risk-free interest rate r because $e^{-r(T-t)}\varphi(-i,t;T)$ does not depend on r, thus $\mathcal{R}_t^{\mathrm{Net}}=\mathcal{R}_t^{M}$ holds. This completes the proof. □

4 Numerical results

In this section, we use numerical examples to illustrate the effect of the VIX-linked fee on the insurer's GMMB liabilities. First, we analyze various combinations of fair VIX-linked fees using formulas developed in Section 3. We then illustrate the effect of the VIX-linked fee structure on current and future liabilities.

4.1 Market and VA assumptions

In order to present useful numerical results, we need an accurate and recent calibration of the Heston model. For the purposes of this paper, a joint $\mathbb{P}-$measure and $\mathbb{Q}-$measure calibration would be ideal, since we use both measures to obtain relevant numerical results. Such a joint calibration of the Heston model is challenging and the literature on the topic is sparse. The methods developed for such an estimation are often complex and/or require high-frequency data. For example, Garcia, Lewis, Pastorello, and Renault (2011) use 5-minute returns and daily option data over 10 years.

One-factor stochastic volatility models such as the Heston model do not typically provide a very good fit to equity index data. At least two factors, or jumps, are needed to adequately reproduce market features (see Garcia, Lewis, Pastorello, and Renault (2011) and references therein). Nonetheless, the Heston model is used here for its analytic tractability. It allows for faster calculations and is sufficient to replicate the general tendencies of the market. We use the Heston model to assess the influence of the VIX-linked fee on VA liabilities, in the same mindset that the Black-Scholes model, a simplification of actual equity index dynamics, was considered in MacKay, Augustyniak, Bernard, and Hardy (2015).

For the numerical examples presented later in this section, we use a parameter set available in the literature. The one we choose is obtained in Guillaume and Schoutens (2010). Although it is only fitted to option prices (thus resulting in a $\mathbb{Q}$-measure parameter set), it has the advantage of using relatively recent data[f]. For the purpose of calculations under the real-world measure, we select equity and volatility risk premium parameters that lead to reasonable market dynamics. Since we recognize that the parameters chosen may not fit market data perfectly, we test the sensitivity of our results to the volatility risk premium parameter λ.

Thus, throughout this section, unless otherwise indicated, we use the parameters in Table 1, obtained by Guillaume and Schoutens (2010).

Parameter	Value
κ	0.5780
$\bar{V}$	0.0518
V_0	0.0225
σ	0.2446
ρ	-0.8872

Table 1: Market Parameters

In addition, we assume $r = 0.02$ and $\mu = 0.04$. To obtain $\mathbb{P}-$measure parameters for the process V_t, we assume[g] $\lambda = -0.25$, which gives $\kappa^* = 0.828$ and $\bar{V}^* = 0.0362$. The parameter λ is chosen to obtain realistic values for κ^* and $\bar{V}^*$, and it is negative following evidence presented in Bakshi and Kapadia (2003). In Section 4.4, we study the sensitivity of our results to λ to assess the effect of this assumption. Unless otherwise indicated, we consider a GMMB with the specifications given in Table 2:

In Table 2, δ is the guaranteed roll-up rate, and is generally assumed to be less than r. The value we choose for the investment fund management fee, c^{inv}, is motivated by

[f]The parameters we use are from Guillaume and Schoutens (2010)'s full calibration to S&P500 option prices as of 18/07/2007.

[g]In comparison, Aït-Sahalia and Kimmel (2007) obtain $\kappa^* = 5.07$ and $\bar{V}^* = 0.0457$, while Garcia, Lewis, Pastorello, and Renault (2011) get $\kappa^* = 0.173$ and $\bar{V}^* = 0.809$. Our κ^* falls between these two calibrations, while our $\bar{V}^*$ is close to the first one.

Parameter	Value
F_0	100
T	10
G	$F_0 e^{\delta T}$
c^{inv}	0.0075

Table 2: VA Parameters

recent data on average expense ratios for mutual funds in the United States.[h] In the next section, we perform sensitivity tests to assess the effect of this parameter on the fair fee rates.

4.2 Fair fee

In this section, we study fair fee structures, as defined by (21) in Section 3. We show that for a fair fee couple $(\bar{c}^*, m^*)$, a lower base fee $\bar{c}^*$ is associated with a higher multiplier m^*.

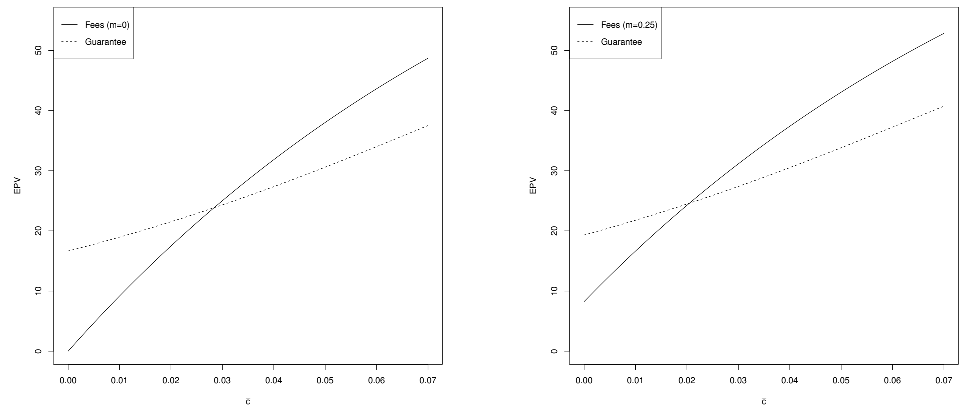
4.2.1 Fair $\bar{c}$ for fixed m

Figure 1 illustrates the present value of the financial guarantee and of the total fee income as a function of the base fee $\bar{c}$ for fixed values $m = 0$ and $m = 0.25$, when the investment fund management fee rate is 0.75%. Note that $m = 0$ is equivalent to the fixed percentage fee case. For a given value of m, the fair base fee rate $\bar{c}^*$ is the value at which both curves intersect. While both values depend on the fee rate, the fee income increases significantly with m, leading to a lower fair base fee rate $\bar{c}^*$. This is intuitive: when a part of the rider fee income is based on the VIX, the base fee can be lower than when the fee rate is constant, and result in a similar income.

Table 3 presents the fair base fee $\bar{c}^*$ for different values of m and c^{inv}. As expected, for a fixed multiplier m and investment fund fee c^{inv}, the fair base fee increases with the guaranteed roll-up rate δ. The fair base fee $\bar{c}^*$ is a decreasing function of the multiplier m. Thus, as the VIX-linked part of the rider fee increases, the fixed part of the rider fee does not need to be as high for the guarantee to be covered.

The investment fund management fee has a significant impact on the fair fee structure. As c^{inv} increases, the base fee rate $\bar{c}^*$ must also increase to fund the financial guarantee. Indeed, the investment fund fee reduces the total return on the VA account, which in turn increases the value of the VA financial guarantee. The latter must be financed by a higher rider fee. The results of Table 3 show that this is particularly true for higher guaranteed roll-up rates δ. Therefore, insurers must pay particular attention to management fees when selecting the investment choices offered to the policyholder.

[h] Data found at `/www.ici.org/pressroom/news/16_news_trends_expenses`.



(a) $m = 0$ (b) $m = 0.25$

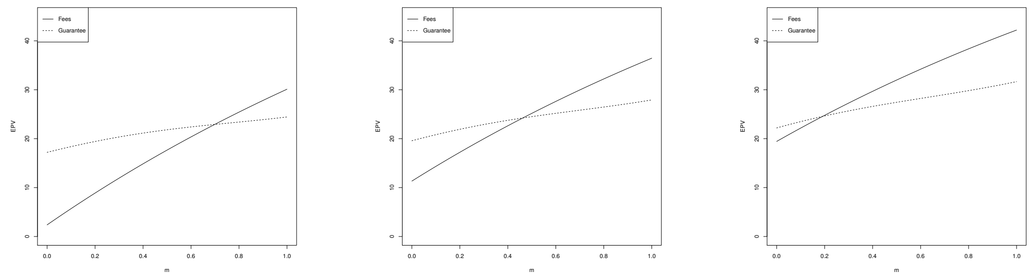
Figure 1: Expected present value of the financial guarantee for $\delta = 0$ and the rider fee income as a function of $\bar{c}$, with $c^{inv} = 0.75\%$.

	$m = 0$	$m = 0.15$	$m = 0.30$	$m = 0.45$
$c^{inv} = 0\%$				
$\delta = 0.0\%$	2.2613%	1.8026%	1.2938%	0.7361%
$\delta = 0.5\%$	2.7544%	2.3032%	1.7899%	1.2191%
$\delta = 1.0\%$	3.4564%	3.0133%	2.4908%	1.9002%
$c^{inv} = 0.5\%$				
$\delta = 0.0\%$	2.6243%	2.1695%	1.6569%	1.0905%
$\delta = 0.5\%$	3.2584%	2.8104%	2.2902%	1.7071%
$\delta = 1.0\%$	4.2451%	3.8030%	3.2670%	2.6574%
$c^{inv} = 0.75\%$				
$\delta = 0.0\%$	2.8389%	2.3856%	1.8704%	1.2990%
$\delta = 0.5\%$	3.5706%	3.1234%	2.5985%	2.0082%
$\delta = 1.0\%$	4.7921%	4.3478%	3.8021%	3.1815%
$c^{inv} = 1.00\%$				
$\delta = 0.0\%$	3.0817%	2.6295%	2.1111%	1.5343%
$\delta = 0.5\%$	3.9400%	3.4926%	2.9618%	2.3637%
$\delta = 1.0\%$	5.5292%	5.0782%	4.5203%	3.8898%

Table 3: Fair base fee $\bar{c}^*$.

4.2.2 Fair m for fixed $\bar{c}$

In this section we solve for the fair multiplier m^* for different values of the base fee $\bar{c}$. Figure 2 shows that the present value of the financial guarantee and the fee income are both increasing functions of m. Indeed, a higher m translates into a higher fee, which in turn depletes the underlying fund faster and increases the value of the guarantee.



(a) $\bar{c} = 0.25\%$ (b) $\bar{c} = 1.25\%$ (c) $\bar{c} = 2.25\%$

Figure 2: Expected present value of the financial guarantee for $\delta = 0$ and the rider fee income as a function of m, with $c^{inv} = 0.75\%$.

Table 4 presents the fair multiplier m^* for different values of δ and $\bar{c}$, when $c^{inv} = 0.75\%$. Note that for δ fixed, if we set $\bar{c}$ higher than the fair fee rate $\bar{c}^*$ for $m = 0$ (see the second column of Table 3), the fair multiplier m does not exist. In fact, the curves would intersect at $m^* < 0$, but we only allow positive values for the multiplier m.

δ	$\bar{c} = 0.25\%$	$\bar{c} = 0.75\%$	$\bar{c} = 1.25\%$	$\bar{c} = 1.75\%$	$\bar{c} = 2.25\%$
0.0%	0.6985	0.5834	0.4623	0.3328	0.1912
0.5%	0.8411	0.7353	0.6259	0.5116	0.3903
1.0%	1.0577	0.9598	0.8604	0.7589	0.6548

Table 4: Fair multiplier m^* for different values of $\bar{c}$, with $c^{inv} = 0.75\%$.

The fair VIX-linked fee rates for $\delta = 0$ are presented in Table 5 for different values of V_t. Note that $V_t = 0.15$ (the last row) gives $\sqrt{V_t} = 0.3873$, which is a level of volatility reached during the most recent financial crisis[i]. At this level, the rider fee rate becomes very high as soon as it is linked to the VIX index, even when the multiplier is as low as 0.2 (second to last column of Table 5). This motivates a potential cap on the rider fee rate, to keep the product marketable and to reduce the fee drag (i.e. the negative net return on the VA account resulting from a high fee) that could happen in periods of high volatility.

m^*	0.6985	0.5834	0.4623	0.3328	0.1912	0.0000
$\bar{c}^*$ (in %)	0.25	0.75	1.25	1.75	2.25	2.84
$V_t = 0.0100$	1.02	1.39	1.75	2.12	2.46	2.84
$V_t = 0.0225$	1.86	2.10	2.32	2.52	2.69	2.84
$V_t = 0.0500$	3.75	3.67	3.56	3.42	3.21	2.84
$V_t = 0.1500$	10.57	9.68	8.08	6.67	5.07	2.84

Table 5: Fair fee rates c_t (in %), $\delta = 0$.

[i]See historical VIX data at `https://fred.stlouisfed.org/series/VIXCLS/`.

In the subsequent sections, we only consider quantities linked to fairly priced VA policies. In other words, the fee parameters $(\bar{c}, m)$ that we consider are always fair. To simplify the notation, we drop the asterisk superscript going forward.

4.3 Effect of the VIX-linked fee on the net liability

4.3.1 Sensitivity of the net liability to V_t

The main objective of the VIX-linked rider fee is to reduce the riskiness of insurers' VA liabilities. In particular, it should reduce the sensitivity of the net liability, when compared to the constant fee rate structure. In this section, we provide a numerical example to analyze the sensitivity of the net liability to the instantaneous volatility.

Recall that the net liability, $\Pi_t^{\mathrm{Net}}(T-t, F_t, V_t)$, defined in (27), takes into account the value of the future fee income. Going forward, we will shorten the notation and denote the net liability at t by Π_t^{Net}. We consider the same contract as in the previous sections ($T = 10$, $\delta = 0$, $c^{inv} = 0.75\%$) and plot the net liability Π_t^{Net} at different times during the life of the contract ($t \in \{0, 2, 8\}$) for various account values ($F_t \in \{80, 100, 120\}$). For $t = 0$, we only study the case $F_0 = 100$, because we assume that the contract starts at the money. We compare the net liability under different fee structures, namely the fixed fair fee rate ($(\bar{c}, m) = (2.8389\%, 0)$) and the VIX-linked fair fee rates given in Table 4.

Figure 3 presents the net liability at inception of the VA contract for different values of V_0 and for different fee structures. Recall that any fair fee structure satisfies Definition 3.1, which equivalently means that the net liability at inception is equal to 0, with the model assumptions in Table 1. Hence, the net liability is 0 for any fair fee structure when $V_0 = 0.0225$. Figure 3 shows that when the rider fee is strongly linked to the VIX ($(\bar{c}, m) = (0.25\%, 0.6985)$), the net liability barely increases with V_0. Furthermore, as the base fee $\bar{c}$ increases, so does the slope of the net liability with respect to V_0. In other words, as the rider fee's dependence on the VIX decreases (or when $\bar{c}$ is lower), the sensitivity of the net liability to changes in the instantaneous volatility increases. This observation is in line with the proxy liability analysis made using historical observations in the second part of the white paper (CBOE, 2013b, (Figure 1 on page 6)).

Figure 4 shows that for all combinations of time to maturity and account values under consideration, a lower $\bar{c}$, associated with a higher fee multiplier m (see Table 4), reduces the sensitivity of the net liability Π_t^{Net} to the instantaneous volatility V_t. This effect is more pronounced when the guarantee is out of the money ($F_t > G$), but can also be observed for lower values of F_t. In particular, if the value of the VA account F_t increases significantly shortly after the inception of the contract, a VIX-linked fee can cause the net liability to decrease as V_t increases (see Figure 4(c)). In such a situation, the value of the GMMB is close to zero, and a higher volatility only has a significant impact on the rider fee income.

Compared to other fair fee structures, a higher multiplier m leads to a smaller net liability for higher volatility V_t, since it better matches the liability with future fee income. In fact, a higher instantaneous volatility causes the liability to increase. When the rider fee rate is linked to the VIX, the fee income also increases. Thus, Figure 4 confirms the

CBOE claim that such a fee "reduce[s] the impact of implied volatility changes on reserve and capital costs" (CBOE 2013a, page 2).

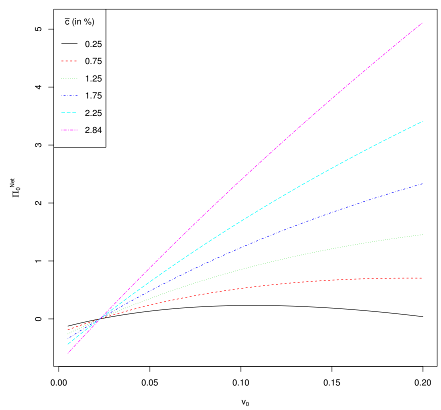
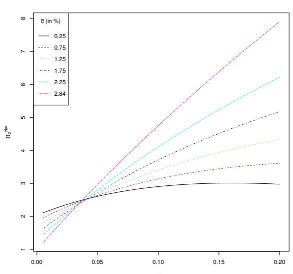
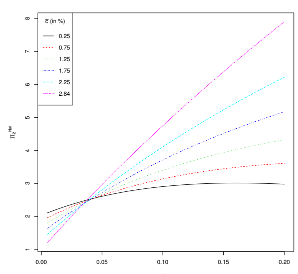


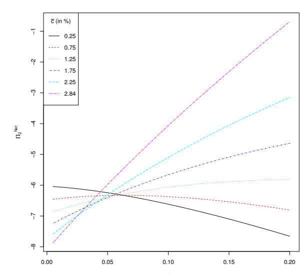
Figure 3: Net liability for fixed and VIX-linked fees, $t = 0$.



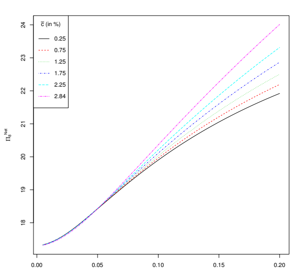
(a) $F_2 = 80$, $t = 2$



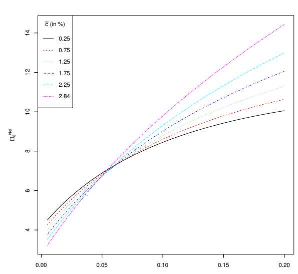
(b) $F_2 = 100$, $t = 2$



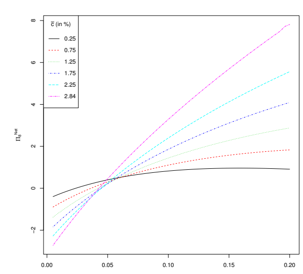
(c) $F_2 = 120$, $t = 2$



(d) $F_8 = 80$, $t = 8$



(e) $F_8 = 100$, $t = 8$



(f) $F_8 = 120$, $t = 8$

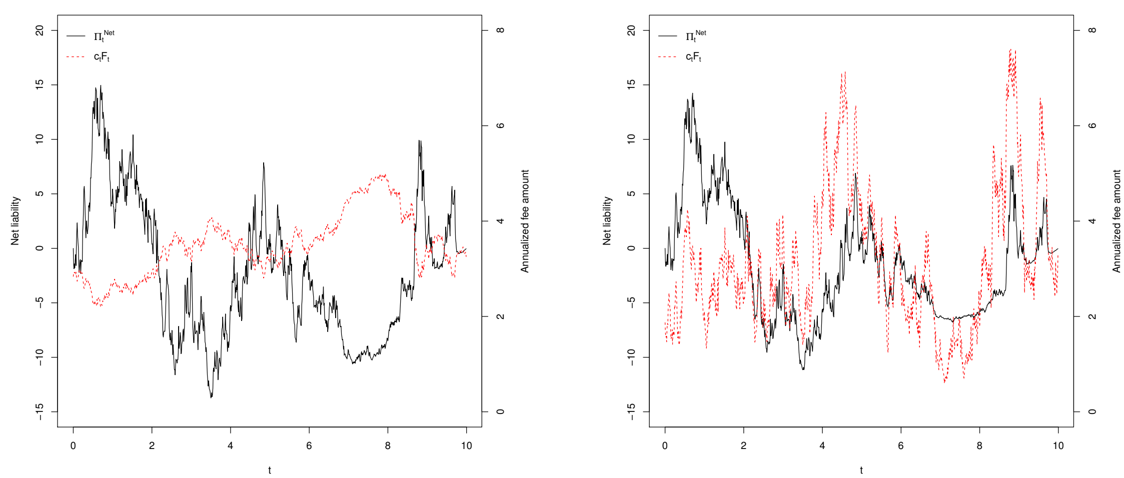
Figure 4: Net liability for fixed and VIX-linked fees, $t = 2$ and $t = 8$.

4.3.2 Alignment of the net liability and the fee income

Here we present a numerical example that illustrates the *improved alignment* of the fee income with the net liability under the VIX-linked fee structure. We consider the same contract as in the previous example ($T = 10$, $\delta = 0$, $c^{inv} = 0.75\%$). We simulate a path of the VA account value and the associated instantaneous volatility process for a 10-year period, using the model presented in Section 2.4 and the parameters given at the beginning of this section.

To obtain the discretized path, we use the drift interpolation approximation to Broadie and Kaya (2006)'s exact scheme (see van Haastrecht and Pelsser (2010)) and consider 200 steps per year. For each time point $t_j = \frac{j}{200}$, with $j \in \{0, \ldots, 200\}$, we calculate the net liability and the annualized fee income using the simulated account value $\widetilde{F}_{t_j}$ and instantaneous volatility $\widetilde{V}_{t_j}$. The net liability is obtained using (27), while the annualized fee income is simply given by $c_{t_j} \widetilde{F}_{t_j}$.

The resulting paths of net liability and fee income for fixed percentage and VIX-linked fee structures are presented in Figure 5. When the fee is set as a fixed percentage ($m = 0$), the fee amount is negatively correlated with the net liability. In particular, the fee amount collected is at its lowest when the liability is high, which is problematic from a risk management point of view. In comparison, the VIX-linked fee structure with $m = 0.6985$ leads to a better alignment of the fee income and the net liability, with the fee income increasing with the liabilities.



(a) $\bar{c} = 2.26\%, m = 0$ (fixed percentage fee) (b) $\bar{c} = 0.25\%, m = 0.2539$ (VIX-linked fee)

Figure 5: Net liability and annualized fee income for a simulated VA contract.

4.4 Effect of the VIX-linked fee on the future net liability

In this section, we analyze the effect of the VIX-linked fee rider on the real-world distribution of the insurer's future net liability. In particular, we are interested in the one-year ahead liability, which can be used to predict the insurer's financial position in one year. We show that, when re-calibration of the model results in a higher long-term mean $\bar{V}$ for the variance process, the VIX-linked fee leads to a significant reduction in the expected future net liability. However, in general, the effect of the VIX-linked fee on the distribution of the one-year ahead liability is not as pronounced. Information on the one-year-ahead liability is of interest in the insurance industry, as it can be used for internal risk management purposes, and is also relevant to regulators for solvency purposes.

The analysis conducted in this section is analogous to the "Stochastic Results" presented in Figures 7 and 8, on page 12 of the CBOE white paper (2013b), which uses an "illustrative set of real-world stochastic scenarios to project one year". They compare VA contracts with constant and VIX-linked fee rates. Their results indicate that the distribution of the one-year ahead liability of a VA with VIX-linked fees has lower standard deviation and 99.5% VaR. The methodology behind this analysis is not detailed, but our goal here is to mirror these results through the real-world distribution of the one-year ahead net liability calculated using the model developed in Section 2.1.

To obtain the $\mathbb{P}$-distribution of the future net liability, we use the dynamics of the account value under the real-world measure, as presented in Section 2.4. We are then interested in the distribution of Π_{t+1}^{Net} conditional on $\mathcal{F}_t$.

An empirical estimate of the distribution of the future net liability is obtained by simulating $M = 10,000$ paths of the account value and of the volatility process up to time $t+1$. Again, the simulations are performed using the drift interpolation approximation to Broadie and Kaya's exact scheme (see van Haastrecht and Pelsser (2010)). We denote the resulting simulated account values and instantaneous volatilities by $\widetilde{F}_{t+1}^{(i)}$ and $\widetilde{V}_{t+1}^{(i)}$, for $i = 1, 2, \ldots, M$. These values are then substituted in (27) to calculate the net liability resulting from each simulated path. Note that without the formulas given in Propositions 3.1 and 3.2, this analysis would require nested simulations.

For different VIX-linked fee structure, the mean, the 95% value-at-risk ($\text{VaR}_{95\%}$) and the 95% expected shortfall ($\text{ES}_{95\%}$) are calculated. The empirical estimates of the risk measures are obtained using the formulas given in Section 9.2 of McNeil, Frey, and Embrechts (2015).

The VA contract we consider throughout this section is the same as in the previous ones ($T = 10$, $\delta = 0$, $c^{inv} = 0.75\%$). Unless otherwise indicated, we use the market parameters given in Table 1.

In Table 6, we present estimated statistics of the one-year ahead future liability at different times throughout the duration of the contract. Although the effect is not always significant, our results show that the VIX-linked fee structure can increase the average one-year ahead liability. Based on the VaR and the expected shortfall, the right tail of the distribution of the one-year ahead liability appears thicker with the VIX-linked fee. This means that in the worst cases, the VIX-linked fee structure increases the liabilities.

This is due to the fact that when the volatility becomes very high, the VIX-linked fee rate increases (see Table 5) and drags the VA account value down. A possible solution to this problem would be to cap the VIX-linked fee rate, as is done in practice. However, with such a cap, closed form expressions for the expected fund value and the liabilities might be impossible to derive. Further work should nonetheless consider fee caps.

m $\bar{c}$ (in %)	0.6985 0.25	0.5834 0.75	0.4623 1.25	0.3328 1.75	0.1912 2.25	0.0000 2.84
			$t = 0$			
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \mid A\right]$	1.59	1.59	1.59	1.59	1.60	1.60
$\widehat{\text{VaR}}_{95\%}$	19.74	19.86	19.99	20.06	20.07	20.03
$\widehat{\text{ES}}_{95\%}$	26.96	26.96	26.96	26.95	26.91	26.79
			$t = 5$			
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \mid A\right]$	6.73	6.55	6.37	6.19	6.02	5.84
$\widehat{\text{VaR}}_{95\%}$	25.16	25.16	25.15	25.07	24.98	24.80
$\widehat{\text{ES}}_{95\%}$	33.07	32.99	32.88	32.76	32.60	32.34
			$t = 8$			
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \mid A\right]$	7.03	6.92	6.81	6.70	6.58	6.43
$\widehat{\text{VaR}}_{95\%}$	28.16	28.12	28.09	27.97	27.87	27.61
$\widehat{\text{ES}}_{95\%}$	37.01	36.87	36.71	36.53	36.29	35.93

Table 6: Mean, VaR and ES of future net liabilities, $A = \{F_t = 100, V_t = 0.0225\}$.

4.4.1 Sensitivity of the results to key parameters

Sensitivity to $\bar{V}$

Figure 4 shows that when V_t is sufficiently high, the VIX-linked fee generally leads to a lower net liability. Therefore, one might expect that the tail risk measures presented in Table 6 decrease with $\bar{c}$. In other words, a rider fee that is more strongly linked to the VIX should improve the insurer's financial situation in the worst-case scenarios, since those usually occur when market volatility is high. However, the results presented in Table 6 do not confirm this intuition. Instead, the tail risk measures of the one-year-ahead net liability is lower when the fee rate is constant ($m = 0$). Further analysis show that this is due to the particular parameter set we are using. It describes a market where V_t stays relatively low, and therefore the simulated values $\widetilde{V}_{t+\tau}^{(i)}$ very rarely reach the level where the VIX-linked fee has a significant impact. For this reason, the effect of the VIX-linked rider fee on the distribution of the one-year-ahead liability is less obvious.

To assess the impact of the VIX-fee in different volatility regimes, we now carry out the test using different values $\bar{V}$, which is the parameter linked to the long-term mean of the variance process. Higher values of $\bar{V}$ will lead to higher simulated instantaneous volatilities at time $t + 1$, and will show the effect of the VIX-linked fee structure when the long-term market volatility changes. It is important to note here that we perform the analysis using the fair fee couples obtained for our original value $\bar{V} = 0.0518$.

The results are presented in Table 7. For $\bar{V} = 0.08$, the reduction of the expected value of the one-year-ahead liability caused by the VIX-linked fee structure is clear. This means that if the VA contract is priced in a relatively low volatility environment, and that a later model re-calibration results in a higher $\bar{V}$, the VIX-linked fee structure helps to keep the net liability low. The opposite is true if the re-calibrated $\bar{V}$ drops; in that case, the VIX-linked fee structure leads to a liability higher than that in the fixed percentage fee case. This is explained by Figures 3 and 4, which show that for low values of V_t, the net liability with VIX-linked fee becomes higher than its counterpart with the constant rider fee. In fact, low volatility reduces the fee income when the fee structure is linked to the VIX. Nonetheless, regardless of the fee structure, the future net liability is reduced when $\bar{V}$ is low.

In summary, this last numerical example shows that the VIX-linked fee structure makes the future liability *less sensitive* to re-calibration of the long-term volatility term. In other words, under the VIX-linked fee structure, the distribution of the one-year-ahead liability is less affected when the volatility of the underlying index undergoes a long-term change.

m	0.6985	0.5834	0.4623	0.3328	0.1912	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	2.84
$\bar{V} = 0.03$						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \mid A\right]$	5.79	5.09	4.43	3.81	3.22	2.57
$\widehat{\text{VaR}}_{95\%}$	22.87	22.71	22.65	22.51	22.29	22.02
$\widehat{\text{ES}}_{95\%}$	31.64	31.52	31.39	31.23	30.03	30.72
$\bar{V} = 0.08$						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \mid A\right]$	7.13	7.55	7.94	8.33	8.71	9.17
$\widehat{\text{VaR}}_{95\%}$	27.93	27.95	28.01	28.05	28.14	28.23
$\widehat{\text{ES}}_{95\%}$	35.68	35.64	35.60	35.55	35.48	35.34

Table 7: Mean, VaR and ES of future net liabilities at $t = 5$, $A = \{F_t = 100, V_t = 0.0225\}$
.

Sensitivity to $\boldsymbol{\kappa}^*$

In the literature, different calibrations of the Heston model lead to very different values for certain parameters. For example, Aït-Sahalia and Kimmel (2007) obtain $\kappa^* = 5.07$ for the speed of mean reversion parameter. This is significantly higher than the value we used in the previous numerical analysis (under the assumption $\lambda = -0.25$, we have $\kappa^* = 0.828$). It should be noted, however, that Aït-Sahalia and Kimmel (2007) use the arbitrary assumption $\lambda = 0$. Nonetheless, this important difference in the value of the parameter motivates sensitivity analysis. In fact, a high speed of mean-reversion could reduce the efficacy of the VIX-linked fee by making the changes in the volatility only transitory.

Keeping all other parameters as in Table 1, we set $\kappa^* = 2.5$ and $\kappa^* = 4.5$, each time with $\lambda = 0$ (which yields $\bar{V} = 0.0518$) and *re-calculate the fair fee structure for different* *values of* $\bar{c}$. Different statistics pertaining to the resulting distribution of the one-year ahead net liabilities are presented in Table 8.

$\kappa^* = 2.5$						
m	0.6635	0.5735	0.4812	0.3863	0.2882	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	3.61
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \vert A\right]$	8.32	8.29	8.26	8.23	8.20	8.14
$\widehat{\text{VaR}}_{95\%}$	29.88	29.86	29.86	29.82	29.76	29.54
$\widehat{\text{ES}}_{95\%}$	36.94	36.90	36.84	36.77	36.70	36.43
$\kappa^* = 4.5$						
m	0.6554	0.5683	0.4794	0.3884	0.2950	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	3.73
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}} \vert A\right]$	8.51	8.49	8.47	8.46	8.44	8.42
$\widehat{\text{VaR}}_{95\%}$	29.77	29.77	29.75	29.75	29.72	29.55
$\widehat{\text{ES}}_{95\%}$	36.13	36.11	36.08	36.05	36.01	35.82

Table 8: Mean, VaR and ES of future net liabilities, $t = 5$, $\lambda = 0$, $A = \{F_t = 100, V_t = 0.0225\}$.

A first observation is that, compared to the original parameter set, corresponding values of the fair multiplier, m, are lower for low values of $\bar{c}$, but do not decrease as quickly as values of $\bar{c}$ increase. It follows that, both when $\kappa^* = 2.5$ and $\kappa^* = 4.5$, the fair fixed fee rate is higher than in the original parameter set.

While the overall level of the future liabilities is higher with the new parameter set, the distribution of the one-year ahead net liability becomes slightly less sensitive to changes with fee structures as κ^* increases. Lower liabilities are however still observed when the fee rate is fixed.

The parameter κ^* is linked to the speed of mean reversion of the volatility. As κ^* increases, changes in the volatility are more transitory and can have less impact on the distribution of the future net liability. It follows that the impact of the VIX-linked fee is also less significant when κ^* is higher.

Sensitivity to ρ

One of the motivations for the VIX-linked fee is the negative correlation between the volatility of the S&P500 and its value. Market data shows that this measure is unstable through time (see for example Shu and Zhang (2012); Whaley (2009)), which the Heston model fails to take into account. For this reason, it is important to assess the effect of our assumption for the parameter ρ.

Table 9 shows quantities linked to the distribution of the one-year ahead net liability when $\rho = -0.35$, which translates into a weaker correlation between the stock index value and its volatility. The rider fee parameters $(\bar{c}, m)$ are re-calculated so that the fee structure is fair. The results show that the impact of the VIX-linked fee structure on the distribution of the future net liabilities is reduced when the correlation between the stock index value and its volatility is weaker.

m	0.6394	0.5420	0.4420	0.3387	0.2312	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	3.25
$\widehat{\mathbb{E}^{\mathbb{P}}\left[\Pi_{t+1}^{\text{Net}}\|A\right]}$	6.53	6.46	6.40	6.34	6.30	6.22
$\widehat{\text{VaR}}_{95\%}$	23.65	23.71	23.77	23.82	23.85	23.98
$\widehat{\text{ES}}_{95\%}$	30.10	30.15	30.20	30.24	30.28	30.34

Table 9: Mean, VaR and ES of future net liabilities, $t = 5$, $\rho = -0.35$, $A = \{F_t = 100, V_t = 0.0225\}$.

Other parameters

Further sensitivity tests with respect to the underlying fund value F_t and the volatility risk premium parameter λ have been conducted. The results are presented in Tables A1 and A2 in Appendix D. They agree with the previous findings.

The results in Table A1 show that when the guarantee is out of the money ($F_t = 120$), the VIX-linked fee structure can significantly reduce possible profits. This is linked to the misalignment between the fee income and the value of the guarantee. When the latter decreases, the former increases and leads to profits. By reducing the discrepancy between the two quantities, the VIX-linked fee can cut down on the profits.

The absolute value of λ is linked to the magnitude of the volatility risk premium. The results in Table A2 show that a higher volatility risk premium makes the mean of the future net liability more sensitive to changes in the fee structure. When comparing the case $\bar{c} = 0.25\%$ to the constant fee rate case ($m = 0$), the mean of the net liability increases by almost 18% when $\lambda = -1.5$. The same comparison shows an increase of around 10% when $\lambda = 0.5$ and $\lambda = -0.5$. The volatility risk premium has an opposite effect on the tail of the distribution of the future net liability. That is, the sensitivity of tail risk measures to changes in the fee structure (constant or VIX-linked fee) decreases when $|\lambda|$ increases.

5 Conclusion and Future Research

In this paper, we introduced an analytic framework for modeling the VIX-linked fee structure for variable annuities. In particular, the joint evolution of the equity index and the VIX is modeled by the Heston stochastic volatility model, under which the new fee structure leads to the formulation of a new Heston-like model. We developed a closed-form solution to the characteristic function of the log fund value. We also presented expressions for risk-neutral values of the GMMB rider and the Greeks of the associated net liability. We illustrated with numerical examples the effect of the new fee structure on the net liability and on the fee income. Here is a summary of the paper's findings.

1. We show in Section 4.3.1 that with a VIX-linked rider fee, the current net liability of the guarantee is less sensitive to the instantaneous market volatility V_t than when the rider fee is constant. This is in line with Figure 1 of the second part of the CBOE white paper (2013b), which suggests that the insurer's liability under the VIX-linked fee structure is less affected by changes in market volatility, when compared to fixed

percentage fees. This reduced sensitivity, when combined with the new fee structure, can help re-align the fee income with the value of benefit guarantees. We give an example of this improved alignment in Section 4.3.2. One may argue that the VIX-linked fee leads to a more prudent risk management practice, as the income is more likely to cover the liabilities and the higher cost of hedging when market volatility increases.

2. Our numerical examples also demonstrate that a VIX-linked rider fee also reduces the sensitivity of the distribution of the one-year ahead net liability to changes in the long-term mean of the volatility. When compared to the constant rider fee case, this translates into a lower expected future liability when $\bar{V}$ is increased. However, a lower sensitivity also means that the VIX-linked expected future liability decreases less when $\bar{V}$ drops. Therefore, the VIX-linked fee structure reduces the loss of the insurer when a model re-calibration increases $\bar{V}$, at the cost of reducing its profits when long-term market volatility goes down. The VIX-linked fee structure thus appears to be beneficial in terms of protecting the insurer in case of an increase in long-term market volatility.

3. Our numerical results show that, when market volatility becomes very high, the VIX-linked fee can pull the total return of the VA account down. In the worst cases, this can worsen the insurer's financial position, even with the protection that the VIX-linked fee should provide. For this reason, further research should explore caps on the VIX-linked fee. This might reduce the fee drag in extreme cases while providing the re-alignment of the fee income to the value of the guarantee.

As alluded to in the introduction, the current common practice of fixed fee structure can lead to the unintended consequence of adverse selection, where policyholders tend to keep their contract in times of market turmoil, and to surrender under stable market conditions. In addition to increasing the robustness of the net liability to changes in market volatility, the proposed VIX-linked fee structure should reduce the incentive to surrender. While we have analyzed the impact of fee structures on overall insurance liabilities from the insurer's point of view, we have not investigated whether the VIX-linked fee structure can indeed reduce or eliminate adverse selection. Future work should explore whether the VIX-linked fee structure has a significant impact on policyholders' incentives to surrender under various market conditions. The effect of capping the VIX-linked fee, or to revise it only periodically, rather than continuously, should also be considered in future research.

Our results could also be extended to more complex guarantees, such as guaranteed minimum withdrawal benefits. The path-dependent nature of this type of benefit increases the complexity of the derivations and calls for Asian-option style approximations. The performance of the VIX-linked fee structure should also be assessed in more realistic market models, in particular in the presence of jumps in the volatility process (see for example Park (2016)).

References

Aït-Sahalia, Y., and R. Kimmel (2007): "Maximum likelihood estimation of stochastic volatility models," *Journal of Financial Economics*, 83(2), 413–452.

Albrecher, H., P. Mayer, W. Schoutens, and J. Tistaert (2007): "The little Heston trap," *Wilmott Magazine*, January, 83–92.

Bakshi, G., and N. Kapadia (2003): "Delta-hedged gains and the negative market volatility risk premium," *Review of Financial Studies*, 16(2), 527–566.

Bernard, C., M. Hardy, and A. MacKay (2014): "State-dependent fees for variable annuity guarantees," *ASTIN Bulletin*, 44(3), 559–585.

Bernard, C., A. Kolkiewicz, and J. Tang (2016): "Variable Annuities with fees tied to VIX," *working paper, available at http://egrie2016.ucy.ac.cy/wp-content/uploads/2016/09/16_Bernard_2016-06-13-preliminary.pdf.*

Bernard, C., and A. MacKay (2015): "Reducing surrender incentives through fee structure in variable annuities," in *Innovations in Quantitative Risk Management*, pp. 209–223. Springer.

Broadie, M., and O. Kaya (2006): "Exact simulation of stochastic volatility and other affine jump diffusion processes," *Operation Research*, 54(2), 217–231.

CBOE (2013a): "VIX for variable annuities - Part I," *Chicago Board of Options Exchange Working Paper.*

——— (2013b): "VIX for variable annuities - Part II," *Chicago Board of Options Exchange Working Paper.*

Chen, Z., K. Vetzal, and P. A. Forsyth (2008): "The effect of modelling parameters on the value of GMWB guarantees," *Insurance: Mathematics and Economics*, 43(1), 165–173.

Cox, J., J. Ingersoll, and S. Ross (1985): "A theory of the term structure of interest rates," *Econometrica*, 53, 385–407.

Davies, R. B. (1973): "Numerical inversion of a characteristic function," *Biometrika*, 60(2), 415–417.

Delong, Ł. (2014): "Pricing and hedging of variable annuities with state-dependent fees," *Insurance: Mathematics and Economics*, 58, 24–33.

Feng, R. (2014): "A comparative study of risk measures for guaranteed minimum maturity benefits by a PDE method," *North American Actuarial Journal*, 18(4), 445–461.

Feng, R., and Y. Shimizu (2016): "Applications of central limit theorems for equity-linked insurance," *Insurance: Mathematics and Economics*, 69, 138–148.

Feng, R., and H. W. Volkmer (2012): "Analytical calculation of risk measures for variable annuity guaranteed benefits," *Insurance: Mathematics and Economics*, 51(3), 636–648.

——— (2016): "An identity of hitting times and its application to the valuation of guaranteed minimum withdrawal benefit," *Math. Financ. Econ.*, 10(2), 127–149.

Garcia, R., M.-A. Lewis, S. Pastorello, and É. Renault (2011): "Estimation of objective and risk-neutral distributions based on moments of integrated volatility," *Journal of Econometrics*, 160(1), 22–32.

Gatheral, J. (2006): “The volatility surface,” *Wiley Finance Series*.

Gil-Pelaez, J. (1951): “Note on the inversion theorem,” *Biometrika*, 38(3-4), 481–482.

Guillaume, F., and W. Schoutens (2010): “Use a reduced Heston or reduce the use of Heston?,” *Wilmott Journal*, 2(4), 171–192.

Heston, S. (1993): “A closed-form solution for options with stochastic volatility with applications to bond and currency options,” *Review of Financial Studies*, 6(2), 327–343.

Hurd, T., and A. Kuznetsov (2008): “Explicit formulas for Laplace transforms of stochastic integrals„” *Markov Process and Related Fields*, 14, 277–290.

Karatzas, I., and S. Shreve (1991): “Brownian motion and stochastic calculus,” *Graduate Texts in Mathematics. vol. 113, 2nd edn. Springer, New York*.

Liu, Y. (2010): “Pricing and hedging the guaranteed minimum withdrawal benefits in variable annuities,” *Ph.D. Thesis, University of Waterloo*.

Lord, R., and C. Kahl (2010): “Complex logarithms in Heston-like models,” *Mathematical Finance*, 20(4), 671–694.

MacKay, A., M. Augustyniak, C. Bernard, and M. Hardy (2015): “Risk management of policyholder behavior in equity-linked life insurance,” *Journal of Risk and Insurance, forthcoming*.

McNeil, A. J., R. Frey, and P. Embrechts (2015): *Quantitative Risk Management: Concepts, Techniques and Tools: Concepts, Techniques and Tools*. Princeton university press.

Park, Y.-H. (2016): “The effects of asymmetric volatility and jumps on the pricing of VIX derivatives,” *Journal of Econometrics*, 192(1), 313–328.

Rebonato, R. (2005): *Volatility and correlation: the perfect hedger and the fox*. John Wiley & Sons.

Shu, J., and J. E. Zhang (2012): “Causality in the VIX futures market,” *Journal of Futures Markets*, 32(1), 24–46.

van Haastrecht, A., R. Lord, A. Pelsser, and D. Schrager (2009): “Pricing long-dated insurance contracts with stochastic interest rates and stochastic volatility,” *Insurance: Mathematics and Economics*, 45(3), 436–448.

van Haastrecht, A., and A. Pelsser (2010): “Efficient, almost exact simulation of the Heston stochastic volatility model,” *International Journal of Theoretical and Applied Finance*, 13(01), 1–43.

Whaley, R. E. (2009): “Understanding the VIX,” *The Journal of Portfolio Management*, 35(3), 98–105.

Zhang, J., and Y. Zhu (2006): “VIX Futures,” *Journal of Futures Markets*, 26(6), 521–531.

Zhu, Y., and J. Zhang (2007): “Variance term structure and VIX futures pricing,” *International Journal of Theoretical and Applied Finance*, 10(1), 111–127.

A Proof of Proposition 3.1

Proof. Without loss of generality, we consider the case $t = 0$ in this proof for notational brevity. From the Cholesky decomposition, we have $W_t^{(1)} = \rho W_t^{(2)} + \sqrt{1-\rho^2} W_t^{(3)}$, where $W_t^{(2)}$ and $W_t^{(3)}$ are independent standard Brownian motions, and ρ is the correlation coefficient. Integrating both sides of the variance process in (11) from 0 to T gives

$$\int_0^T \sqrt{V_t} dW_t^{(2)} = \frac{V_T - V_0 - \kappa \bar{V} T + \kappa \int_0^T V_t dt}{\sigma}. \tag{29}$$

Let $\gamma := \beta - \bar{V}\alpha$, and from Itô's lemma, we have

$$\begin{aligned}
F_T &= F_0 \exp\left((r-\gamma)T - \left(\alpha + \frac{1}{2}\right)\int_0^T V_t dt + \rho \int_0^T \sqrt{V_t} dW_t^{(2)} + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t} dW_t^{(3)}\right) \\
&= F_0 \exp\left((r-\gamma)T - \left(\alpha + \frac{1}{2}\right)\int_0^T V_t dt + \frac{\rho}{\sigma}\left(V_T - V_0 - \kappa \bar{V} T + \kappa \int_0^T V_t dt\right)\right. \\
&\quad \left. + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t} dW_t^{(3)}\right),
\end{aligned} \tag{30}$$

where in the last equality we have utilized (29). Then the characteristic function of X_T is given by

$$\begin{aligned}
\mathbb{E}[e^{iuX_T}] &= \mathbb{E}\left[\exp\left(iu\left(\frac{\rho\kappa}{\sigma} - \alpha - \frac{1}{2}\right)\int_0^T V_t dt + \frac{iu\rho}{\sigma}V_T + iu\left(r - \gamma - \frac{\rho\kappa\bar{V}}{\sigma}\right)T\right.\right. \\
&\quad \left.\left. - \frac{iu\rho V_0}{\sigma} + iu\sqrt{1-\rho^2}\int_0^T \sqrt{V_t} dW_t^{(3)}\right)\right] \\
&= \mathbb{E}\left[\exp\left(iu\left(\frac{\rho\kappa}{\sigma} - \alpha - \frac{1}{2}\right)\int_0^T V_t dt + \frac{iu\rho}{\sigma}V_T + iu\left(r - \gamma - \frac{\rho\kappa\bar{V}}{\sigma}\right)T\right.\right. \\
&\quad \left.\left. - \frac{iu\rho V_0}{\sigma}\right) \times \mathbb{E}\left[\exp\left(iu\sqrt{1-\rho^2}\int_0^T \sqrt{V_t} dW_t^{(3)}\right) \mid \mathcal{F}_T^V\right]\right] \\
&= \mathbb{E}\left[\exp\left(\left(iu\left(\frac{\rho\kappa}{\sigma} - \alpha - \frac{1}{2}\right) - \frac{(1-\rho^2)u^2}{2}\right)\int_0^T V_t dt + \frac{iu\rho}{\sigma}V_T\right.\right. \\
&\quad \left.\left. iu\left(r - \gamma - \frac{\rho\kappa\bar{V}}{\sigma}\right)T - \frac{iu\rho V_0}{\sigma}\right)\right] \\
&= e^{iu\left(r-\gamma-\frac{\rho\kappa\bar{V}}{\sigma}\right)T - \frac{iu\rho V_0}{\sigma}} \times \\
&\quad \mathbb{E}\left[\exp\left(\left(iu\left(\frac{\rho\kappa}{\sigma} - \alpha - \frac{1}{2}\right) - \frac{(1-\rho^2)u^2}{2}\right)\int_0^T V_t dt + \frac{iu\rho}{\sigma}V_T\right)\right],
\end{aligned} \tag{31}$$

where in the third equality of (31), we have used the fact that $\int_0^T \sqrt{V_t} dW_t^{(3)}$ is normally distributed, i.e. $\mathcal{N}\left(0, \int_0^T V_t dt\right)$, conditional on the filtration $\mathcal{F}_T^V$. This is because $W_t^{(3)}$ is independent of $\mathcal{F}_T^V$.

The joint Laplace transform of V_T and $\int_0^T V_t dt$ can be obtained from Theorem 3.1 of Hurd and Kuznetsov (2008) by some variable substitutions, and we have

$$\mathbb{E}\left[e^{-\eta \int_0^T V_s ds - \lambda V_T}\right] = e^{-\kappa \bar{V} v T} \left(\frac{\gamma_T}{\gamma_T + \lambda - v}\right)^{\frac{2\kappa\bar{V}}{\sigma^2}}$$
$$\times \exp\left(-V_0 \cdot \left(v + \frac{\gamma_T(\lambda - v)}{\gamma_T + \lambda - v} e^{-T\sqrt{\kappa^2 + 2\eta\sigma^2}}\right)\right), \tag{32}$$

where the auxiliary functions are defined as

$$v = -\frac{\kappa}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\kappa^2 + 2\eta\sigma^2};$$
$$\gamma_T = \frac{\frac{2}{\sigma^2}\sqrt{\kappa^2 + 2\eta\sigma^2}}{1 - e^{-T\sqrt{\kappa^2+2\eta\sigma^2}}} = \frac{2\sqrt{\kappa^2 + 2\eta\sigma^2}}{\sigma^2(1 - e^{-T\sqrt{\kappa^2+2\eta\sigma^2}})}. \tag{33}$$

Letting $\eta = iu\left(\alpha + \frac{1}{2} - \frac{\rho\kappa}{\sigma}\right) + \frac{(1-\rho^2)u^2}{2}$ and $\lambda = -\frac{iu\rho}{\sigma}$, we can simplify

$$\begin{aligned} v &= -\frac{\kappa}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\kappa^2 + 2\eta\sigma^2} \\ &= -\frac{\kappa}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\kappa^2 + 2\sigma^2\left(iu\left(\alpha + \frac{1}{2} - \frac{\rho\kappa}{\sigma}\right) + \frac{(1-\rho^2)u^2}{2}\right)} \\ &= -\frac{\kappa}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{(\kappa - i\rho\sigma u)^2 + \sigma^2(i(2\alpha+1)u + u^2)}. \end{aligned} \tag{34}$$

Define $d := \sqrt{(\kappa - i\rho\sigma u)^2 + \sigma^2(i(2\alpha+1)u + u^2)}$. Then we can rewrite $v = (d-\kappa)/\sigma^2$ and

$$\gamma_T = \frac{2\sqrt{\kappa^2 + 2\eta\sigma^2}}{\sigma^2(1 - e^{-T\sqrt{\kappa^2+2\eta\sigma^2}})} = \frac{2d}{\sigma^2(1 - e^{-dT})}. \tag{35}$$

We have $\lambda - v = -iu\rho/\sigma - (d-\kappa)/\sigma^2 = (\kappa - d - iu\rho\sigma)/\sigma^2$. Define $q := \kappa - d - iu\rho\sigma$. Then $\lambda - v = q/\sigma^2$. Define $g := q/(q + 2d) = (\kappa - d - iu\rho\sigma)/(\kappa + d - iu\rho\sigma)$, then

$$\frac{\gamma_T}{\gamma_T + \lambda - v} = \frac{2d}{2d + q - qe^{-dT}} = \frac{1-g}{1 - ge^{-dT}}, \tag{36}$$

and similarly

$$v + \frac{\gamma_T}{\gamma_T + \lambda - v}(\lambda - v)e^{-dT} = \frac{d-\kappa}{\sigma^2} + \frac{q}{\sigma^2}\frac{(1-g)e^{-dT}}{1 - ge^{-dT}}. \tag{37}$$

Then we can finally simplify the characteristic function in (31) as

$$\begin{aligned} \mathbb{E}[e^{iuX_T}] &= \exp\left\{iu(r - \beta + \alpha\bar{V})T - \frac{iu\rho\kappa\bar{V}T}{\sigma} - \frac{iu\rho V_0}{\sigma}\right. \\ &\quad \left. -\kappa\bar{V}T\frac{d-\kappa}{\sigma^2} + \frac{2\kappa\bar{V}}{\sigma^2}\ln\frac{1-g}{1-ge^{-dT}} - V_0\frac{d-\kappa}{\sigma^2} - V_0\frac{q}{\sigma^2}\frac{(1-g)e^{-dT}}{1-ge^{-dT}}\right\} \\ &= \exp\left\{iu(r - \beta + \alpha\bar{V})T + \kappa\bar{V}T\frac{q}{\sigma^2} + \frac{2\kappa\bar{V}}{\sigma^2}\ln\frac{1-g}{1-ge^{-dT}} + \frac{V_0 q}{\sigma^2}\frac{1-e^{-dT}}{1-ge^{-dT}}\right\}. \end{aligned} \tag{38}$$

This completes the proof. □

B Proof of Proposition 3.2

Without loss of generality, we consider the case $t = 0$ in this proof for notational brevity. The formula (19) connecting the distribution function and characteristic function is known in Gil-Pelaez (1951) and Davies (1973).

Introducing a change of measure from $\mathbb{Q}$ to $\widetilde{\mathbb{Q}}$ by a Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{e^{X_T}}{\mathbb{E}^{\mathbb{Q}}[e^{X_T}]} = \frac{e^{X_T}}{\varphi(-i)}.$$

We shall prove in Remark B.1 that $\psi(-i)$ is well-defined. Under this new measure, the characteristic function becomes

$$\widetilde{\varphi}(u) = \mathbb{E}^{\widetilde{\mathbb{Q}}}[e^{iuX_T}] = \frac{\mathbb{E}^{\mathbb{Q}}[e^{X_T+iuX_T}]}{\mathbb{E}^{\mathbb{Q}}[e^{X_T}]} = \frac{\varphi(u-i)}{\varphi(-i)}.$$

Therefore,

$$\Pi_2 = \varphi(-i)\widetilde{\mathbb{Q}}(X_T \leqslant k) = \varphi(-i)\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\widetilde{\varphi}(u)}{iu}\right] du\right\},$$

which leads to the desired formula. This completes the proof. □

Remark B.1. *We provide a short proof that $\varphi(-i)$ is well-defined. Consequently $\varphi(u-i)$ is well-defined for all $u \in \mathbb{R}$. It is easy to see by analytic continuation that $g(\mu, \nu)$ is analytic for*

$$\mu > -\frac{\kappa^2}{2\sigma^2}, \qquad \nu > -\frac{\gamma + \kappa + e^{-\gamma T}(\gamma - \kappa)}{\sigma^2(1 - e^{-\gamma T})}. \tag{39}$$

Therefore, $\varphi(-i)$ is well-defined if we can show that

$$\mu = \alpha - \frac{\rho\kappa}{\sigma} + \frac{\rho^2}{2}, \qquad \nu = -\frac{\rho}{\sigma}$$

both satisfy the inequalities (39)*. The first inequality is satisfied because $\alpha > 0$ and*

$$\frac{\rho^2}{2} - \frac{\rho\kappa}{\sigma} + \frac{\kappa^2}{2\sigma^2} = \frac{1}{2}\left(\rho - \frac{\kappa}{\sigma}\right)^2 \geqslant 0.$$

As a consequence of the first inequality, we have $\gamma > 0$. Since $\sigma > 0$, then

$$\frac{\gamma + \kappa + e^{-\gamma T}(\gamma - \kappa)}{\sigma^2(1 - e^{-\gamma T})} = \frac{2\gamma}{\sigma(1 - e^{-\gamma T})} - \frac{\gamma - \kappa}{\sigma} > \frac{\gamma + \kappa}{\sigma}. \tag{40}$$

Again, because of the first inequality, we have $\gamma > |\rho\sigma - \kappa|$. It follows immediately that

$$\frac{\gamma + \kappa}{\sigma} > \frac{|\rho\sigma - \kappa| + \kappa}{\sigma} \geqslant \rho. \tag{41}$$

Multiplying the combined inequality of (40) *and* (41) *by $-1/\sigma$ yields the second inequality.*

C Proof of Proposition 3.3

Proof. Here we first aim to prove the following identity

$$\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c_u^{tot}F_u du\right] = F_t - \mathbb{E}_t\left[e^{-r(T-t)}F_T\right]. \tag{42}$$

First, denote $\widetilde{F}_t = e^{-rt}F_t$ and $\widetilde{S}_t = e^{-rt}S_t$, so that

$$\begin{aligned} d\widetilde{F}_t &= -c_t^{tot}\widetilde{F}_t dt + \sqrt{V_t}\widetilde{F}_t dW_t^{(1)}, \\ d\widetilde{S}_t &= \sqrt{V_t}\widetilde{S}_t dW_t^{(1)}. \end{aligned}$$

If we assume that no fees were paid out of the account from time t to maturity, i.e. the account return would be equal to S_T/S_t on the period $(t,T]$, then the discounted expectation of the account value at maturity is given by F_t. If fees are paid from the account, then the discounted expected value of the account at maturity is $\mathbb{E}_t\left[e^{-r(T-t)}F_T\right]$. Thus the right-hand side of (42) is the difference between the two aforementioned values:

$$\begin{aligned} \mathbb{E}_t\left[e^{-r(T-t)}\left(F_t\frac{S_T}{S_t} - F_T\right)\right] &= \frac{F_t}{S_t}\mathbb{E}_t\left[e^{-r(T-t)}S_T\right] - \mathbb{E}_t\left[e^{-r(T-t)}F_T\right] \\ &= F_t - \mathbb{E}_t\left[e^{-r(T-t)}F_T\right]. \end{aligned}$$

We can show that this expression is also equal to the left-hand side of (42):

$$\begin{aligned} &\mathbb{E}_t\left[F_t e^{-r(T-t)}\frac{S_T}{S_t} - F_T e^{-r(T-t)}\right] = F_t\mathbb{E}_t\left[\frac{\widetilde{S}_T}{\widetilde{S}_t} - \frac{\widetilde{F}_T}{\widetilde{F}_t}\right] \\ &= F_t\mathbb{E}_t\left[\frac{\widetilde{S}_t}{\widetilde{S}_t} + \int_t^T \frac{d\widetilde{S}_u}{\widetilde{S}_t} - \frac{\widetilde{F}_t}{\widetilde{F}_t} - \int_t^T \frac{d\widetilde{F}_u}{\widetilde{F}_t}\right] \\ &= F_t\left\{\mathbb{E}_t\left[\int_t^T \sqrt{V_u}\left(\frac{\widetilde{S}_u}{\widetilde{S}_t} - \frac{\widetilde{F}_u}{\widetilde{F}_t}\right)dW_u\right] + \mathbb{E}_t\left[\int_t^T c_u^{tot}\frac{\widetilde{F}_u}{\widetilde{F}_t}du\right]\right\}. \end{aligned}$$

The first expectation in the equation above vanishes because the Itô integral is a martingale. Thus we have established the identity (42). One can also prove the identity using the Dynkin's formula (c.f. (Feng and Volkmer, 2016, Section 4.3)).

From the identity (42), we can simplify the calculation of the left hand side of (21) as

$$\begin{aligned} \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c_u F_u du\right] &= \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c_u^{tot}F_u du\right] - \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c^{inv}F_u du\right] \\ &= F_t - \mathbb{E}_t[e^{-r(T-t)}F_T] - \int_t^T e^{-r(u-t)}c^{inv}\mathbb{E}_t\left[F_u\right]du. \end{aligned} \tag{43}$$

Note that $\mathbb{E}_t[F_u], u \geqslant t$ can be easily represented in terms of the characteristic function in Proposition 3.1 as $\mathbb{E}_t[F_u] = \varphi(-i,t;u), u \geqslant t$. Then we have

$$\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}c_u F_u du\right] = F_t - e^{-r(T-t)}\varphi(-i,t;T) - c^{inv}\int_t^T e^{-r(u-t)}\varphi(-i,t;u)du. \tag{44}$$

This completes the proof. □

D Additional results from numerical examples

m	0.6985	0.5834	0.4623	0.3328	0.1912	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	2.84
$F_t = 80$						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\mathrm{Net}} \vert A\right]$	17.02	17.06	17.09	17.10	17.09	17.01
$\widehat{\mathrm{VaR}}_{95\%}$	37.54	37.52	37.49	37.33	37.31	37.12
$\widehat{\mathrm{ES}}_{95\%}$	44.46	44.36	44.25	44.11	43.94	43.67
$F_t = 120$						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\mathrm{Net}} \vert A\right]$	-0.04	-0.42	-0.77	-1.08	-1.34	-1.51
$\widehat{\mathrm{VaR}}_{95\%}$	15.29	15.27	15.28	15.26	15.29	15.17
$\widehat{\mathrm{ES}}_{95\%}$	23.30	23.24	23.18	23.10	23.00	22.85

Table A1: Mean, VaR and ES of future net liabilities, $t = 5$, $A = \{F_t = 100, V_t = 0.0225\}$.

m	0.6985	0.5834	0.4623	0.3328	0.1912	0.0000
$\bar{c}$ (in %)	0.25	0.75	1.25	1.75	2.25	2.84
$\lambda = 0.5$ ($\kappa^* = 0.078$, $\bar{V}^* = 0.3839$)						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\mathrm{Net}} \vert A\right]$	7.98	7.81	7.64	7.47	7.33	7.19
$\widehat{\mathrm{VaR}}_{95\%}$	30.54	30.39	30.23	30.03	29.85	29.60
$\widehat{\mathrm{ES}}_{95\%}$	39.38	39.19	38.99	38.76	38.51	38.14
$\lambda = -0.5$ ($\kappa^* = 1.078$, $\bar{V}^* = 0.0278$)						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\mathrm{Net}} \vert A\right]$	6.26	6.44	6.26	6.09	5.92	5.73
$\widehat{\mathrm{VaR}}_{95\%}$	24.24	24.24	24.22	24.16	24.00	23.88
$\widehat{\mathrm{ES}}_{95\%}$	32.87	32.80	32.72	32.61	32.47	32.22
$\lambda = -1.5$ ($\kappa^* = 2.078$, $\bar{V}^* = 0.0144$)						
$\widehat{\mathbb{E}}^{\mathbb{P}}\left[\Pi_{t+1}^{\mathrm{Net}} \vert A\right]$	5.67	5.50	5.33	5.16	5.00	4.81
$\widehat{\mathrm{VaR}}_{95\%}$	19.34	19.43	19.48	19.47	19.48	19.40
$\widehat{\mathrm{ES}}_{95\%}$	26.91	26.94	26.96	26.95	26.91	26.79

Table A2: Mean, VaR and ES of future net liabilities, $t = 5$, $A = \{F_t = 100, V_t = 0.0225\}$.