Optimal Surrender Policy
for Variable Annuity Guarantees

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Abstract

This paper proposes a technique to derive the optimal surrender strategy for a variable annuity (VA) as a function of the underlying fund value. This approach is based on splitting the value of the VA into a European part and an early exercise premium following the work of Kim and Yu (1996) and Carr, Jarrow, and Myneni (1992). The technique is first applied to the simplest VA with GMAB (path-independent benefits) and is then shown to be possibly generalized to the case when benefits are path-dependent. Fees are paid continuously as a fixed percentage of the fund value. Our approach is useful to investigate the impact of path-dependent benefits on surrender incentives.

Keywords: Variable annuities, Optimal surrender, GMMB, GMSB.

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1 Introduction

A variable annuity (VA) is a unit-linked insurance product offering a variety of financial guarantees. Usually the policyholder pays an initial premium to initiate the contract. This premium is invested in a mutual fund selected by the policyholder. Many types of guarantees and options can be added to the contract (for more details, see Hardy (2003)). In this paper we will focus on a variable annuity contract that guarantees a minimum amount at maturity. This type of VA is referred to as a guaranteed minimum accumulation benefit (GMAB) (see Bauer, Kling, and Russ (2008)). We study two cases, one with a point-to-point guarantee linked to the terminal value of the fund and a guarantee linked to the average value of the fund.

In most cases, the policyholder can choose to lapse the VA contract and receive a surrender benefit, which is less than or equal to the value accumulated in the underlying account. For example Kling, Ruez, and Ruß (2011) show that unexpected lapses represent a significant risk for the insurer. In fact, selling a VA contract is expensive and insurers typically reimburse the expense incurred using the fees paid during the first years of the policy. If the policyholder lapses before the initial expenses are reimbursed, the insurer may experience a loss. Even if they occur later during the life of the contract, lapses can be very expensive.

For this reason, the option to lapse the contract needs to be taken into account and priced in the contract. This is not necessarily simple since assumptions must be made on the surrender behavior of policyholders. Different approaches have been taken in the literature, ranging from a simple deterministic surrender rate to more sophisticated models, like De Giovanni (2010)'s rational expectation and Li and Szimayer (2010)'s limited rationality. Most of these approaches assume that the policyholder cannot calculate the exact risk-neutral value of the contract, and that he may be influenced by exogenous factors.

Another way to approach the surrender problem is to assume the policyholder is perfectly rational and will surrender their contract only when it is optimal to do so from a financial perspective. In this approach, the surrender option is analogous to an American option that can be surrendered at any time before maturity (see Grosen and Jørgensen (2000)). Assuming that the policyholder is perfectly rational leads to an upper bound for the price of the surrender option and gives a lot of insight on the intrinsic value of the options in the VA contract. Although it is not necessarily used to obtain the final price of the VA contract, it can be very useful to assess the risk of optimal

surrender. Furthermore, while there are also other reasons why a VA contract might be surrendered, some policyholders tend to act in a rational way. In their study, Knoller, Kraut, and Schoenmaekers (2011) investigate various hypothesis for an early surrender. For one, they analyze the moneyness of the option as a reason to lapse the VA. This is similar to optimally surrendering the contract when the maturity benefit is out-of-the-money. They find that financial literacy leads to a higher sensitivity towards the moneyness. They also examine other reasons for lapsing the contract such as financial needs of the policyholder or better opportunity costs in times of rising market interest rates.

While it is common practice for insurers to charge a constant fee rate as a percentage of the fund value to cover the maturity benefit and other financial guarantees, many authors assume that all the fees are included in the initial premium (see, for example, Grosen and Jørgensen (2002), Bacinello (2003a), Bacinello (2003b), Siu (2005), Bacinello, Biffis, and Millossovich (2009), Bacinello, Biffis, and Millossovich (2010), Bernard and Lemieux (2008)). However, as pointed out for instance by Bauer, Kling, and Russ (2008), Milevsky and Salisbury (2001) and Bernard, Hardy, and MacKay (2013), a fee paid as a regular constant percentage of the fund might increase the incentive to surrender the contract before the maturity if the fund value is high. This is due to the mismatch between the amount of the fee and the value of the guarantee option. When the fund value is high, the guarantee at maturity is deep out-of-the-money and it is unlikely that the policyholder will make use of the option at expiration. However, since the fee is charged as a percentage of the fund value, the amount of the fee is large. This mismatch represents an incentive to surrender the VA contract and should be taken into account especially when the policyholder is assumed to lapse optimally. Milevsky and Salisbury (2001) discuss this issue, and argue that surrender charges are necessary to hedge VA contract appropriately. In fact, in most VA contracts sold in the industry, early surrenders trigger a surrender charge and the policyholder does not receive the full value accumulated in the underlying fund. This is especially true in the first years of the contract. This surrender charge has many purposes, one of which is to reduce the incentive to surrender. It is also in place to recover the high expenses related to the sale of the VA contract. While this fee does give the policyholder an incentive to remain in the contract, there are many situations where it is optimal to surrender, even after taking the surrender charge into account.

In this paper we investigate the optimal surrender strategy for a variable annuity contract with constant fee rate paid as a percentage of the fund and a GMAB feature. We first consider a simple point-to-point guarantee and derive an integral representation

for the price of the contract, which can be solved to compute the optimal surrender boundary. To do so, we use no-arbitrage arguments presented, among others, by Kim and Yu (1996) and Carr, Jarrow, and Myneni (1992). This technique originally designed for vanilla call options can be extended to more complex path-dependent payoffs linked for example to the average. Our objective is to illustrate a general technique to compute the optimal surrender strategy for a possibly path-dependent contract. This technique may help to understand the effect of complex path-dependent benefits on surrender incentives and could be useful to reduce the surrender option value by modifying the type of benefits offered and justifying the need for path-dependent benefits. This is in contrast with the recent proposal by Bernard, Hardy, and MacKay (2013) to influence the surrender behaviour by charging a state-dependent fee structure, instead of charging a constant fee rate[1].

The paper is organized as follows. In Section 2 we state the setting. The optimal surrender policy is derived in Section 3. Section 4 extends this method to path-dependent payoffs. In Section 5 we apply these results to numerical examples and analyze the sensitivity of the boundary with respect to a range of parameters. Section 6 concludes.

2   Setting

Consider a variable annuity contract with a guaranteed minimum accumulation benefit G_T at maturity T. This accumulation benefit is computed as $G_T = Ge^{gT}$ where the guaranteed rate g satisfies $g < r$. Let F_t denote the underlying accumulated fund value of the variable annuity at time t. We assume that the insurance company charges a constant fee c for the guarantee, which is continuously withdrawn from the accumulated fund value F_t. Furthermore, we assume that the policyholder pays a single premium to initiate the contract. The insurer then invests this premium in the fund or index that was chosen by the policyholder. We denote this underlying fund or index by S_t and assume that it follows a geometric Brownian motion. Therefore, its dynamics under the risk-neutral measure $\mathbb{Q}$ are given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

where r is the risk-free interest rate, $\sigma > 0$ the constant volatility and W_t the Brownian motion. We denote by $\mathcal{F}_t$ the natural filtration associated with this Brownian motion.

[1]In the model of Bernard, Hardy, and MacKay (2013) the policyholder only pays the fee as long as the fund value stays underneath a certain barrier.

In this case, the stock price at time $u > t$ given the stock price at time t has a lognormal distribution and is explicitly given by

$$S_u = S_t e^{(r - \frac{\sigma^2}{2})(u-t) + \sigma(W_u - W_t)}$$

In this paper, we are only concerned with the pricing of the surrender option and as such, we can treat the whole problem under the risk-neutral measure. This choice is also motivated by the use of no-arbitrage arguments in the derivation of the expression for the surrender option. It is based on the assumption that investors optimize over all possible surrender strategies and will choose to surrender optimally from a financial perspective. As investors do not always act optimally, our derivations lead to an upper bound on the price of the surrender option.

The following results (2) and (3) will be useful to derive the results of this paper. Since the insurance company continuously takes out a percentage fee c of the fund value, we have the following relationship between S_u and F_u at any time u

$$F_u = e^{-cu} S_u = F_t e^{(r - c - \frac{\sigma^2}{2})(u-t) + \sigma(W_u - W_t)}. \tag{2}$$

Therefore, the conditional distribution of $F_u | F_t$ for $u > t$ is a lognormal distribution with mean $\ln(F_t) + (r - c - \frac{\sigma^2}{2})(u - t)$ and variance $\sigma^2(u - t)$. Hence, the risk-neutral transition density function of F_u at time $u > t$ given F_t equates to

$$f_{F_u}(x | F_t) = \frac{1}{\sqrt{2\pi\sigma^2(u-t)}x} e^{-\frac{[\ln(\frac{x}{F_t}) - (r - c - \frac{\sigma^2}{2})(u-t)]^2}{2\sigma^2(u-t)}}, \quad x > 0. \tag{3}$$

Note that in this paper we restrain ourselves to the case when the underlying follows a geometric Brownian motion, which presents a simple closed expression for its transition density. However, the method we present here can easily be extended to more general market models. We discuss this point briefly in the conclusion.

2.1 Fair Fee for the European Benefit

Let us assume in this paragraph that the VA cannot be surrendered early and let c be the fee charged by the insurer between 0 and T. Note that the fund value at time T depends on this fee. We denote by F_T^c the value at T of the fund given that the fee charged during $[0, T]$ is equal to c and by $\phi(F_\bullet^c, T)$ the payoff at maturity T which may

depend on the path of the fund denoted by $F_\bullet^c$. If the fee c is fair (for the European benefit), we denote it by c^* and it fulfills

$$F_0 = \mathbb{E}[e^{-rT}\max(\phi(F_\bullet^{c^*}, T), G_T)], \tag{4}$$

where F_0 is the lump sum paid initially by the policyholder net of initial expenses and management fees. This fee c^* exists and is unique. To compute this fair fee, it is always possible to use Monte Carlo techniques. However when the distribution of $\phi(F_\bullet^c, T)$ is known, an analytical formula may be derived, which subsequently can be solved for c^*. For example when $\{X_t\}_{t\in[0,T]}$ is a Markov process with $X_T|X_t \sim \mathcal{LN}(M_t, V_t)$ (a lognormal distribution with parameters $\mathbb{E}[X_T|X_t] = M_t$ and $\text{var}(X_T|X_t) = V_t$), then $\mathbb{E}[\max(X_T, G)]$ can be computed as

$$\mathbb{E}[\max(X_T, G)|\mathcal{F}_t] = e^{M_t + \frac{V_t}{2}}\Phi\left(\frac{-\ln(G) + M_t + V_t}{\sqrt{V_t}}\right) + G\Phi\left(\frac{\ln(G) - M_t}{\sqrt{V_t}}\right) \tag{5}$$

We omit the proof as it is a rather standard computation. The expression (5) can be used to compute the European value of the VA in a Black Scholes setting when $\phi(F_\bullet^{c^*}, T) = F_T^{c^*}$, which is the simplest benefit: a GMAB on the terminal fund value payable at time T (Section 3). We can then solve for the fair fee in (4). It will also be applied when $\phi(F_\bullet^{c^*}, T)$ is the geometric average of the fund value in Section 4.

2.2 Surrender Option

We now assume that the policyholder is allowed to surrender the policy at any time $t \in [0, T)$ for a surrender benefit equal to

$$(1 - \kappa_t)F_\bullet^c$$

where κ_t is a penalty percentage charged for surrendering at time t. This is consistent with the modeling of surrender charges in Milevsky and Salisbury (2001). A standard penalty is typically decreasing over time. Examples of penalty functions are given in Palmer (2006).

In the absence of a surrender penalty ($\forall t, \kappa_t = 0$), we will see in the numerical analysis in Section 5 that the optimal surrender boundary is decreasing as a function of c. This result is intuitive: if the fee c charged on the fund is high, the policyholder has a larger incentive to surrender the contract when the guarantee is out of the money, because

he is paying more for it.[2] This observation means that it may be difficult to pay for the surrender benefit by withdrawing a higher fixed percentage of the fund. Indeed if, for example, it is optimal to surrender when $F_t > 125$ when $c = 1\%$, then by charging $c = 2\%$ it might be optimal to surrender when $F_t > 100$. Increasing the fee c to take into account the surrender benefit increases the value of the surrender option. Alternatives include the possibility to charge for this benefit initially as a lump payment or to design a sufficiently high surrender penalty to decrease the incentive to surrender. This point is already present in the analysis of Milevsky and Salisbury (2001). It is clear that when κ_t is sufficiently high then it is never optimal to surrender at time t.

For simplicity, throughout the paper, we assume that κ_t is exponentially decreasing and equal to $1 - \exp(-\kappa(T - t))$ so that the surrender benefit is equal to

$$e^{-\kappa(T-t)}F_\bullet^c, \tag{6}$$

for $\kappa < c$. For example when the surrender benefit at time t is $e^{-\kappa(T-t)}F_t^c$, then the inequality $\kappa < c$ ensures that it can be optimal to surrender the VA for a sufficiently high value of the fund F_t^c. The continuation value of the contract at time t is indeed always strictly greater than $F_t^c e^{-c(T-t)}$ because the policyholder will receive $\max(F_T^c, G_T)$ at time T and thus at least the fund F_T^c. At time t, the value of receiving F_T^c at time T is given by $\mathbb{E}[F_T^c e^{-r(T-t)}|\mathcal{F}_t] = e^{-c(T-t)}F_t^c$. By assuming that $\kappa < c$, we ensure that for any fixed time $t \in [0, T)$, there exists a fund value high enough that the surrender benefit is worth more than the maturity benefit so that surrendering the policy might become optimal.

3   Derivation of the optimal exercise boundary

This section presents the technique used to derive the optimal surrender boundary. As already mentioned earlier it can sometimes be optimal for the policyholder to surrender the contract before the maturity T because the fee c is charged as a percentage of the fund value. Thus, assuming the fund value is sufficiently high, the fee paid for the guarantee would be too high compared to the actual value of the guarantee. This mismatch leads to an optimal early surrender of the variable annuity.

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[2]In other words at a given time, the higher c the larger is the future fees to pay before the maturity whereas the final benefit is decreasing in c, so the gap between the future benefit associated with the guarantee option and the future expected fees remaining to be paid increases and thus the incentive to surrender increases as well.

Consider the variable annuity contract from Section 2 with a payoff of $\max(F_T, G_T)$ at maturity T, where we assume that c is given and thus omit the exponent c in the value of the fund at time t. If the contract is surrendered early at time $t < T$, the policyholder receives the accumulated fund value F_t reduced by the penalty fees so that the surrender benefit is given by $e^{-\kappa(T-t)}F_t$ (particular case of (6)). Let B_t denote the value of the optimal exercise boundary at time t, i.e. if the fund value crosses this value from below, it is optimal for the policyholder to lapse the contract and receive the amount B_t.[3]

In order to derive the price and the boundary conditions we use the same technique as Kim and Yu (1996) and Carr, Jarrow, and Myneni (1992) and we decompose the price at time t of the VA denoted by $V(F_t, t)$ into a European part and an early exercise premium. To understand the intuition behind this approach, consider the following trading strategy which "converts" the American part of the option, i.e. the surrender option, into the corresponding European option and the early exercise premium. We know that the price of the VA at time $t < T$ along the exercise boundary is equal to $e^{-\kappa(T-t)}F_t$ because the surrender benefit at t is $e^{-\kappa(T-t)}F_t$. Moreover, $B_0 > F_0$ because otherwise it would not be optimal for the policyholder to buy the VA at time 0 for a price F_0. We neglect all transaction costs.

Assume that the policyholder has bought the VA at time $t = 0$. Now whenever the fund value crosses the optimal exercise boundary from below, he exercises the option and surrenders the contract. And whenever the fund value crosses the boundary from above, he buys back the VA contract (given that the boundary is exactly equal to the value of the VA by definition). Any profits resulting from this trading strategy constitute the early exercise premium from the additional surrender benefit in the VA. So assume that at time t the fund value F_t crosses the optimal exercise boundary from below. The policyholder surrenders the contract and receives $e^{-\kappa(T-t)}F_t = e^{-\kappa(T-t)}e^{-ct}S_t$ which he instantaneously invests in the stock S_t. However, since S_t is not subject to the guarantee fee c, S_t outperforms F_t. Therefore, in the case that the fund value crosses the exercise boundary from above, say at time $u > t$, the value of the contract on the boundary is $e^{-\kappa(T-u)}F_u$, the policyholder only needs to pay $e^{-\kappa(T-u)}F_u$ to re-enter, that is $e^{-\kappa(T-u)}e^{-cu}S_u = e^{-\kappa T}e^{-(c-\kappa)u}S_u < e^{-\kappa(T-t)}e^{-ct}S_u$ (because $c - \kappa > 0$). The profit from this strategy is the early exercise premium. A formal derivation is given in the proof of Proposition 3.1 below.

Proposition 3.1. *The benefit associated with the early exercise of the surrender option*

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[3]It is proved in Appendix A that the optimal exercise region is of the form $\{F_t > B_t\}$, in other words the optimal surrender behavior is based on a threshold strategy where optimal exercise is driven by the value of the underlying fund crossing a barrier.

between $[t, t+dt]$ for an infinitesimal time step dt equates to $h(t) = e^{-\kappa(T-t)}(c-\kappa)F_t$.

Proof. Assume the variable annuity is surrendered at time t. Then the policyholder receives an amount of $e^{-\kappa(T-t)}F_t = e^{-\kappa(T-t)}e^{-ct}S_t$, which is invested in the asset S_t. In order to buy it back at time $t+dt > t$, he only needs $e^{-\kappa(T-(t+dt))}F_{t+dt} = e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt}$. Therefore, consider the following decomposition of the amount received at time t:

$$
\begin{aligned}
e^{-\kappa(T-t)}e^{-ct}S_t &= e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_t + e^{-\kappa T}S_t(e^{-(c-\kappa)t} - e^{-(c-\kappa)(t+dt)}) \\
&= e^{-\kappa(T-(t+dt))}e^{-c(t+dt)}S_t + e^{-\kappa(T-t)}e^{-ct}S_t(1 - e^{-(c-\kappa)dt}) \quad (7)
\end{aligned}
$$

The first addend is the amount invested in the asset S_t that is needed to re-enter the contract at time $t+dt$ (in other words, it is the no-arbitrage price of $e^{-\kappa(T-(t+dt))}e^{-c(t+dt)}S_{t+dt}$ paid at time $t+dt$). The second addend is the amount that needs to be siphoned off and is invested in the risk-free asset. This decomposition is going to be the key step in generalizing this proof to more general benefits (see Section 4 for an example of path-dependent benefit).

Now we can look at what happens to this portfolio after we perform the time step from t to $t+dt$. We use the first order approximation to approximate $e^{-(c-\kappa)dt}$ and e^{rdt}. Then the right hand side of (7) becomes

$$
\begin{aligned}
e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt} &+ e^{-\kappa T}e^{-(c-\kappa)t}S_t e^{rdt}(1 - e^{-(c-\kappa)dt}) \\
&= e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt} + e^{-\kappa T}e^{-(c-\kappa)t}S_t(1 + rdt)(c-\kappa)dt + o(dt) \\
&= e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt} + e^{-\kappa T}e^{-(c-\kappa)t}S_t(c-\kappa)dt + o(dt) \\
&= e^{-\kappa(T-(t+dt))}F_{t+dt} + e^{-\kappa(T-t)}(c-\kappa)F_t dt + o(dt)
\end{aligned}
$$

The first part of the expression is the cost of buying back the variable annuity. Then the policyholder is left with the benefit of early exercise of $h(t) := e^{-\kappa(T-t)}(c-\kappa)F_t$. $\square$

Using Proposition 3.1 and the trading strategy explained above we are now able to derive a pricing formula for the variable annuity contract with a surrender benefit similarly to Carr, Jarrow, and Myneni (1992).

Theorem 1. *Let $V(F_t, t)$ denote the price at time t of the variable annuity with guarantee G_T at maturity and a surrender benefit equal to the accumulated fund value with some penalty $\kappa > 0$, $e^{-\kappa(T-t)}F_t$. Then $V(F_t, t)$ can be decomposed into a corresponding European part $v(F_t, t)$ and an early exercise premium $e(F_t, t)$*

$$V(F_t, t) = v(F_t, t) + e(F_t, t), \quad (8)$$

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where

$$\begin{cases} v(F_t,t) = e^{-c(T-t)}F_t\Phi\big(d_1(F_t,G_T,T,t)\big) + e^{-r(T-t)}G_T\Phi\big(d_2(F_t,G_T,T,t)\big), \\ e(F_t,t) = e^{-\kappa T}(c-\kappa)F_te^{ct}\int_t^T e^{-(c-\kappa)u}\Phi\big(d_1(F_t,B_u,u,t)\big)du, \end{cases} \quad (9)$$

and $\Phi(x)$ is the standard normal distribution function with d_1 and d_2 defined as

$$\begin{cases} d_1(x,y,T,t) := \dfrac{\ln(\frac{x}{y})+(r-c+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\[2mm] d_2(x,y,T,t) := \sigma\sqrt{T-t}-d_1(x,y,T,t). \end{cases} \quad (10)$$

Proof. At first we prove the formula for the European part $v(F_t,t)$ of the VA. Since $F_T|F_t \sim \mathcal{LN}(\ln(F_t)+(r-c-\frac{\sigma^2}{2})(T-t),\sigma^2(T-t))$, we can use (5) to calculate the European part of the VA. Define d_1 and d_2 as in (10). Then it follows that

$$v(F_t,t) = e^{-r(T-t)}\left[F_te^{(r-c)(T-t)}\Phi\left(\frac{-\ln(G_T)+\ln(F_t)+(r-c+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$

$$\left. +G_T\Phi\left(\frac{\ln(G_T)-\ln(F_t)-(r-c-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right],$$

and we find (9). Secondly, we prove the formula for the early exercise premium $e(F_t,t)$. Define $\widetilde{\mu}(x) := \ln(F_t)+(r-c-\frac{\sigma^2}{2})(x-t)$ and $\widetilde{\sigma}^2(x) := \sigma^2(x-t)$. From Proposition 3.1, the benefits of an early exercise amounts to $e^{-\kappa(T-u)}(c-\kappa)F_u du$ whenever the fund F_u is above the optimal exercise boundary B_u at any time $u > t$. Therefore, the early exercise premium at $t < T$ can be calculated by the following formula

$$e(F_t,t) = \int_t^T e^{-r(u-t)}\int_{B_u}^{\infty} e^{-\kappa(T-u)}(c-\kappa)xf_{F_u}(x|F_t)dxdu$$

$$\overset{(3)}{=} (c-\kappa)\int_t^T e^{-\kappa(T-u)}e^{-r(u-t)}\int_{B_u}^{\infty}x\frac{1}{\sqrt{2\pi\sigma^2(u-t)}x}e^{-\frac{[\ln(\frac{x}{F_t})-(r-c-\frac{\sigma^2}{2})(u-t)]^2}{2\sigma^2(u-t)}}dxdu$$

$$= (c-\kappa)\int_t^T e^{-\kappa(T-u)}e^{-r(u-t)}\int_{B_u}^{\infty}\frac{1}{\sqrt{2\pi\widetilde{\sigma}^2(u)}}e^{-\frac{[\ln(x)-\widetilde{\mu}(u)]^2}{2\widetilde{\sigma}^2(u)}}dxdu$$

$$\overset{y=\ln(x)}{=} (c-\kappa)\int_t^T e^{-\kappa(T-u)}e^{-r(u-t)}\int_{\ln(B_u)}^{\infty}\frac{1}{\sqrt{2\pi\widetilde{\sigma}^2(u)}}e^{-\frac{[y-\widetilde{\mu}(u)]^2}{2\widetilde{\sigma}^2(u)}}e^ydydu$$

$$= (c-\kappa)\int_t^T e^{-\kappa(T-u)}e^{-r(u-t)}\int_{\ln(B_u)}^{\infty}\frac{1}{\sqrt{2\pi\widetilde{\sigma}^2(u)}}e^{-\frac{[y-(\widetilde{\mu}(u)+\widetilde{\sigma}^2(u))]^2}{2\widetilde{\sigma}^2(u)}}\underbrace{e^{\widetilde{\mu}(u)+\frac{\widetilde{\sigma}^2(u)}{2}}}_{=F_te^{(r-c)(u-t)}}dydu$$

$$
\begin{aligned}
&= (c-\kappa)F_t \int_t^T e^{-\kappa(T-u)}e^{-c(u-t)}\left[1 - \Phi\left(\frac{\ln(B_u) - (\widetilde{\mu}(u) + \widetilde{\sigma}^2(u))}{\widetilde{\sigma}(u)}\right)\right]du \\
&= (c-\kappa)F_t \int_t^T e^{-\kappa(T-u)}e^{-c(u-t)}\Phi\left(\frac{\ln(\frac{F_t}{B_u}) + (r - c + \frac{\sigma^2}{2})(u-t)}{\sigma\sqrt{u-t}}\right)du \\
&= (c-\kappa)F_t \int_t^T e^{-\kappa(T-u)}e^{-c(u-t)}\Phi\big(d_1(F_t, B_u, u, t)\big)du.
\end{aligned}
$$

The expression for the early exercise premium in (9) follows. $\qquad\square$

Theorem 1 provides a way to calculate the price of a VA with surrender benefit. However, since the early exercise premium depends on the optimal exercise boundary B_t, one needs to compute it first. In the following we derive the optimal exercise boundary condition in analogy to Kim and Yu (1996).

First, note that the value of B_T at maturity equals to $B_T = G_T$. We also know that along the exercise boundary it holds

$$
V(F_t, t) = e^{-\kappa(T-t)}F_t = B_t.
$$

Thus, by formula (8) and (9) we have

$$
\begin{aligned}
B_t &= v(F_t, t) + e(F_t, t) \\
&= e^{-c(T-t)}B_t e^{\kappa(T-t)}\Phi(d_1(B_t e^{\kappa(T-t)}, G_T, T, t)) + e^{-r(T-t)}G_T\Phi(d_2(B_t e^{\kappa(T-t)}, G_T, T, t)) \\
&\quad + (c-\kappa)B_t e^{(c-\kappa)t}\int_t^T e^{-(c-\kappa)u}\Phi\left(d_1\left(B_t e^{\kappa(T-t)}, B_u, u, t\right)\right)du.
\end{aligned}
\tag{11}
$$

This integral equation can be used to compute the optimal exercise boundary B_t. Observe, however, that in order to equate B_t the optimal exercise boundary for future times must be known. Since it holds that $B_T = G_T$ at expiration, we work backwards through time to recursively recover the optimal exercise boundary. Because formula (11) does not have an analytic solution, numerical integration schemes must be used. Practically this is done by dividing the interval $[0, T]$ into n equidistant subintervals $0 = t_0 < t_1 < ... < t_n = T$ where times t_i, $i = 0, .., n$, represent the only possible early exercise times. Define $g(u) := e^{-(c-\kappa)u}\Phi(d_1(B_t e^{\kappa(T-t)}, B_u, u, t))$. Then, the integral in (11) is approximated by

$$
I(k) = \frac{T}{n}\sum_{i=1}^{k-1} g(t_{n-i}), \quad k = 1, .., n.
\tag{12}
$$

Note, that at time t_{n-1} the early exercise premium $I(1)$ is equal to zero because there is no possibility for the policyholder to surrender the option in the last interval. Therefore, the premium has to be zero.

Proposition 3.2 (Derivation of the optimal exercise boundary). *The following backward procedure allows to derive the exercise boundary approximately.*

- $B_{t_n} = B_T = G_T$.

- *Recursively, for $k = 1..n$, compute $I(k)$ in (12) to approximate the right part of (11) and solve the following equation for the only unknown $B_{t_{n-k}}$*

$$
\begin{aligned}
B_{t_{n-k}} &= e^{-c(T-t_{n-k})} B_{t_{n-k}} e^{\kappa(T-t_{n-k})} \Phi(d_1(B_{t_{n-k}} e^{\kappa(T-t_{n-k})}, G_T, T, t_{n-k})) \\
&\quad + e^{-r(T-t_{n-k})} G_T \Phi(d_2(B_{t_{n-k}} e^{\kappa(T-t_{n-k})}, G_T, T, t_{n-k})) + (c-\kappa) B_{t_{n-k}} e^{(c-\kappa)t_{n-k}} I(k).
\end{aligned}
$$

The method described in this Section 3 is straightforward to extend to any path-independent payoff for which $\phi(F_\bullet^c(T), T) = \ell(F_T^c, T)$ for some function $\ell(\cdot)$. In the next section we illustrate how to derive the optimal exercise boundary when $\phi(F_\bullet^c(T), T)$ is possibly path-dependent, that is it depends on the path $(F_t)_{t\in[0,T]}$.

4 Path-dependent payoff

In this section, we consider a path-dependent design of the payoff of the variable annuity. The example that we study is based on the following payoff $\phi(F_\bullet^c) = \max(G_T, Y_T)$ computed as the maximum of the geometric average Y_T of the fund value at time T and of the guarantee G_T at time T. The geometric average Y is defined as

$$
Y_t = \exp\left(\frac{1}{t} \int_0^t \ln F_s \, ds\right). \tag{13}
$$

Our goal is twofold. First we illustrate a general method to derive the optimal surrender strategy when there are path-dependent benefits. Second, we want to understand the impact of Asian benefits on the surrender incentive in VAs.

We need a few preliminary results. Defining the geometric average of the index similarly as

$$
\widetilde{Y}_t = \exp\left(\frac{1}{t} \int_0^t \ln S_s \, ds\right) \tag{14}
$$

gives us the following relation between Y_t and $\widetilde{Y}_t$ at any time t

$$
Y_t = e^{-\frac{ct}{2}} \widetilde{Y}_t. \tag{15}
$$

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An important difference with the setting of Section 3 is that this payoff is path-dependent since Y_t includes all values of F_s for times $s \in [0, t]$. We assume that the surrender benefit at time t is now also path-dependent and equal to

$$e^{-\kappa(T-t)}Y_t, \tag{16}$$

where κ is sufficiently small so that it can still be optimal to surrender the policy. In particular, throughout the section, we have the following assumption.

Assumption 4.1. *The parameters r, c and κ are such that*

- $\kappa < \frac{r+c+\frac{\sigma^2}{6}}{2}$, *and*

- $c < r - \frac{\sigma^2}{6}$.

Note that this assumption is not very restrictive. In fact, with a fee rate that would fail to meet the second point of Assumption 4.1, the policy would hardly be marketable.

In the same setting as described in Section 2, the conditional distribution of Y_u to $\mathcal{F}_t$ for $u > t$ can be computed as well. Precisely, the conditional distribution of $\widetilde{Y}_u|(\widetilde{Y}_t, S_t)$ follows a lognormal distribution

$$\widetilde{Y}_u|(\widetilde{Y}_t, S_t) \sim \mathcal{LN}\left(\frac{t}{u}\ln\widetilde{Y}_t + \frac{u-t}{u}\ln S_t + \frac{r - \frac{\sigma^2}{2}}{2u}(u-t)^2 \; , \; \frac{\sigma^2}{3u^2}(u-t)^3\right).$$

This result is known and can be found for example in Hansen and Jørgensen (2000). Using the relationships (15) and (2), it is easy to show from the previous result on S_t and $\widetilde{Y}_t$ that

$$Y_u|(Y_t, F_t) \sim \mathcal{LN}\left(\frac{t}{u}\ln Y_t + \frac{u-t}{u}\ln F_t + \frac{r - c - \frac{\sigma^2}{2}}{2u}(u-t)^2 \; , \; \frac{\sigma^2}{3u^2}(u-t)^3\right) \tag{17}$$

Therefore, the conditional distribution function of Y_u given (Y_t, F_t) is known, similarly to the conditional distribution of $F_u|F_t$ in (3) which was key in the derivation of the early exercise premium for path-dependent benefits.

Using a similar trading strategy as in Section 3, we compute the early exercise premium of the variable annuity with Asian benefits and are able to prove the following proposition.

Proposition 4.1. *The benefit associated with the early exercise of the surrender option between $[t, t+dt]$ for an infinitesimal time step dt equates to*

$$h(t, Y_t, F_t) = e^{-\kappa(T-t)}Y_t\left(r - \kappa + \frac{1}{t}\ln\left(\frac{Y_t}{F_t}\right)\right),$$

when at time t, it is optimal to surrender with (Y_t, F_t).

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Proof. The proof is in the same spirit as the proof of Proposition 3.1 for path-independent benefits. At the optimal boundary, the value of the VA is exactly equal to the surrender benefit (16), therefore

$$V(Y_t, F_t, t) = e^{-\kappa(T-t)}Y_t$$

At time $t + dt$, the value of the contract at the exercise boundary is

$$V(Y_{t+dt}, F_{t+dt}, t + dt) = e^{-\kappa(T-t-dt)}Y_{t+dt}$$

Assume that the VA is surrendered at time t, then the policyholder receives $e^{-\kappa(T-t)}Y_t$, we now have to compute how much is gained by staying out of the contract between t and $t + dt$. The main difficulty is to find a trading strategy at time t to ensure that we are able to re-enter the contract at $t + dt$ and to measure the profit from this strategy needed in the calculation of the early exercise premium.

Let us compute at time t the no-arbitrage value of $e^{-\kappa(T-t-dt)}Y_{t+dt}$. To do so, consider $u > t$ and compute first

$$\mathbb{E}[e^{-r(u-t)}Y_u|\mathcal{F}_t] = e^{-r(u-t)} \exp\left(\frac{t}{u}\ln Y_t + \frac{u-t}{u}\ln F_t + \frac{r - c - \frac{\sigma^2}{2}}{2u}(u-t)^2 + \frac{\sigma^2(u-t)^3}{6u^2}\right)$$

using the conditional distribution of $Y_u|(Y_t, F_t)$. For $u = t + dt$, we find that

$$\mathbb{E}[e^{-rdt}Y_{t+dt}|\mathcal{F}_t] = e^{-rdt}\exp\left(\frac{t}{t+dt}\ln Y_t + \frac{dt}{t+dt}\ln F_t + \frac{r - c - \frac{\sigma^2}{2}}{2(t+dt)}dt^2 + \frac{\sigma^2 dt^3}{6(t+dt)^2}\right).$$

After removing terms negligible against dt

$$\mathbb{E}[e^{-rdt}e^{-\kappa(T-t-dt)}Y_{t+dt}|\mathcal{F}_t] = e^{-\kappa(T-t-dt)}e^{-rdt}\exp\left(\left(1 - \frac{dt}{t}\right)\ln Y_t + \frac{dt}{t}\ln F_t + o(dt)\right),$$

which can be further simplified into

$$\mathbb{E}[e^{-rdt}e^{-\kappa(T-t-dt)}Y_{t+dt}|\mathcal{F}_t] = e^{-\kappa(T-t)}Y_t - e^{-\kappa(T-t)}Y_t\left(r - \kappa + \frac{1}{t}\ln\left(\frac{Y_t}{F_t}\right)\right)dt + o(dt).$$

At time t, the policyholder receives $e^{-\kappa(T-t)}Y_t$. Note the following decomposition,

$$e^{-\kappa(T-t)}Y_t = e^{-\kappa(T-t-dt)}\mathbb{E}[e^{-rdt}Y_{t+dt}|\mathcal{F}_t] + e^{-\kappa(T-t)}Y_t\left(r - \kappa + \frac{1}{t}\ln\left(\frac{Y_t}{F_t}\right)\right)dt + o(dt)$$

One can invest $Y_t e^{-\kappa(T-t)} - Y_t e^{-\kappa(T-t)}\left(r - \kappa + \frac{1}{t}\ln\left(\frac{Y_t}{F_t}\right)\right)dt$ at time t in the delta hedging strategy that generates $e^{-\kappa(T-t)}Y_{t+dt}$ at time $t+dt$. The remainder is left in a bank account at time t, so that the early exercise premium between t and $t + dt$ can be computed as $h(t, Y_t, F_t)$ in Proposition 4.1. $\qquad\square$

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Note that it seems that the early exercise premium can be negative. This is actually not the case as if it is optimal to surrender at time t, then it means that one cannot get better when waiting for another dt, therefore

$$Y_t e^{-\kappa(T-t)} \geqslant \mathbb{E}[e^{-rdt} Y_{t+dt} e^{-\kappa(T-t-dt)} | \mathcal{F}_t],$$

and thus $h(t, Y_y, F_t) \geqslant 0$ at any time t when it is optimal to surrender with (Y_t, F_t).

Proposition 4.2. *Let F_t denote the fund value process given in (2) and Y_t the geometric average based on F_t given in (13). Then, for $u > t$,*

$$Y_u | (Y_t, F_t, F_u = f) \sim \mathcal{LN}\left(M_f, \widehat{V}_{u,t}\right), \tag{18}$$

where $\begin{cases} M_f := M_{Y_u | Y_t, F_t, F_u = f} = \frac{t}{u} \ln Y_t + \frac{1}{2} \frac{u-t}{u} \ln F_t + \frac{u-t}{2u} \ln f, \\ \widehat{V}_{u,t} := V_{Y_u | Y_t, F_t, F_u = f} = \frac{\sigma^2}{12u^2}(u-t)^3. \end{cases}$

Proof. Conditionally on (Y_t, F_t), we have that $(\ln(Y_u), \ln(F_u))$ is a bivariate normal distribution. Thus $\ln(Y_u) | (\ln(F_u), F_t, Y_t)$ is normally distributed with mean $M_{Y_u | Y_t, F_t, F_u}$ and variance $V_{Y_u | Y_t, F_t, F_u}$. To compute the conditional moments of $X|Y$ where $X = \ln Y_u | F_t, Y_t$ and $Y = \ln F_u | F_t, Y_t$ for $u > t$ we first compute

$$\begin{cases} \mathbb{E}[X] = \mathbb{E}[\ln Y_u | F_t, Y_t] = \frac{t}{u} \ln Y_t + \frac{u-t}{u} \ln F_t + \frac{r-c-\frac{\sigma^2}{2}}{2u}(u-t)^2 \\ \mathbb{E}[Y] = \mathbb{E}[\ln F_u | F_t, Y_t] = \mathbb{E}[\ln F_u | F_t] = \ln(F_t) + (r - c - \frac{\sigma^2}{2})(u-t) \\ \mathrm{Var}[X] = \mathrm{Var}[\ln Y_u | F_t, Y_t] = \frac{\sigma^2}{3u^2}(u-t)^3 \\ \mathrm{Var}[Y] = \mathrm{Var}[\ln F_u | F_t, Y_t] = \mathrm{Var}[\ln F_u | F_t, Y_t] = \sigma^2(u-t) \\ \mathrm{cov}[X,Y] = \mathrm{cov}[\ln F_u, \ln Y_u | F_t, Y_t] = \frac{\sigma^2}{2} \frac{(u-t)^2}{u} \\ \mathrm{corr}[X,Y] = \frac{\sqrt{3}}{2} \end{cases} \tag{19}$$

using (3) and (17) for the conditional means and variances. The only missing element is the covariance. From (13), recall that $Y_u = Y_t^{\frac{t}{u}} e^{\frac{1}{u} \int_t^u \ln(F_s) ds}$. It is thus clear that

$$\mathrm{cov}[X,Y] = \mathrm{cov}\left[\ln F_u, \frac{t}{u} \ln Y_t + \frac{1}{u} \int_t^u \ln F_s ds | F_t, Y_t\right] = \mathrm{cov}\left[\ln F_u, \frac{1}{u} \int_t^u \ln F_s ds | F_t, Y_t\right]$$

Using the linearity of the covariance

$$\mathrm{cov}[X,Y] = \frac{1}{u} \int_t^u \mathrm{cov}\left[\ln F_u, \ln F_s | F_t, Y_t\right] ds$$

where we are left with the computation of $\mathrm{cov}\left[\ln F_u, \ln F_s | F_t, Y_t\right]$ for $t \leqslant s \leqslant u$. It is clear that $\mathrm{cov}\left[\ln F_u, \ln F_s | F_t, Y_t\right] = \sigma^2 \mathrm{cov}\left[(W_u - W_t), (W_s - W_t) | F_t, Y_t\right] = \sigma^2 \mathrm{cov}\left[W_{u-t}, W_{s-t} | F_0, Y_0\right] =$

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$\sigma^2 \min(u - t, s - t)$. Integrating over s gives the desired result. Then using the inputs in (19) and the well-known conditional moments of a bivariate normal distribution

$$
\begin{aligned}
M_{Y_u|Y_t,F_t,F_u} &= \mathbb{E}(X) + \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(Y)}(Y - \mathbb{E}(Y)) \\
V_{Y_u|Y_t,F_t,F_u} &= (1 - \rho^2)\,\mathrm{var}(X),
\end{aligned}
$$

where $\rho = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}}$. The claim follows and we have that $Y_u|Y_t, F_t, F_u = f$ is distributed according to a LogNormal distribution with these moments. $\qquad\square$

We can now state a similar result for the early exercise premium with Asian benefits as we derived in Section 3.

Theorem 2. *Let $V^g(Y_t, F_t, t)$ denote the price at time t of the variable annuity with guarantee G_T at maturity and a surrender benefit equal to the accumulated geometric average $e^{-\kappa(T-t)}Y_t$, and suppose that Assumption 4.1 holds. Then $V^g(Y_t, F_t, t)$ can be decomposed into a corresponding European part $v^g(Y_t, F_t, t)$ and an early exercise premium $e^g(Y_t, F_t, t)$*

$$
V^g(Y_t, F_t, t) = v^g(Y_t, F_t, t) + e^g(Y_t, F_t, t), \tag{20}
$$

where

$$
\begin{aligned}
v^g(Y_t, F_t, t) &= e^{-r(T-t)}e^{M_t^g + \frac{V_t^g}{2}}\Phi\left(\frac{-\ln(G_T) + M_t^g + V_t^g}{\sqrt{V_t^g}}\right) + e^{-r(T-t)}G_T\Phi\left(\frac{\ln(G_T) - M_t^g}{\sqrt{V_t^g}}\right), \\
e^g(Y_t, F_t, t) &= e^{-\kappa T}e^{rt}\int_t^T e^{u(\kappa - r)}e^{\frac{\widehat{V}_{u,t}}{2}}Y_t^{\frac{t}{u}}F_t^{\frac{u-t}{2u}}\,\mathbb{E}\left[k(u, F_u, t)\right] du
\end{aligned}
$$

where M_t^g and V_t^g are the conditional moments of $\ln(Y_T)|(Y_t, F_t)$ given in (17), F_u is LogNormal with density $f_{F_u}(f|F_t)$ in (3) and where

$$
k(u, f, t) = f^{\frac{u-t}{2u}}\left(\Phi\left(\frac{H_u(B_u(f), f)}{\sqrt{\widehat{V}_{u,t}}}\right)\left(\frac{H_u(f, f)}{u} + r - \kappa\right) + \frac{\sqrt{\widehat{V}_{u,t}}}{u\sqrt{2\pi}}\,e^{-\frac{1}{2}\frac{H_u(B_u(f),f)^2}{\widehat{V}_{u,t}}}\right)
$$

with $H_u(x, f) = M_f + \widehat{V}_{u,t} - \ln(x)$ and M_f and $\widehat{V}_{u,t}$ are the conditional moments of $Y_u|(Y_t, F_t, F_u = f)$ for $u > t$ given in Proposition 4.2.

Proof. At first we prove the formula for the European part $v^g(Y_t, F_t, t)$ of the VA. Since $Y_T|(Y_t, F_t) \sim \mathcal{LN}(M_t^g, V_t^g)$, we can use (5) to calculate the European part of the VA. $v^g(Y_t, F_t, t)$ in (20) follows immediately. Secondly, we prove the formula for the early exercise premium

$e^g(Y_t, F_t, t)$. Performing a similar substitution as in the derivation of the early exercise premium in Section 3 we get

$$e^g(Y_t, F_t, t) = \int_t^T e^{-r(u-t)} \int_0^\infty \int_{B_u(f)}^\infty h(u, y, f) f_{Y_u}(y|Y_t, F_t, F_u = f) dy f_{F_u}(f|F_t, Y_t) df du \qquad (21)$$

with $h(u, y, f)$ given in Proposition 4.1 and where the rationale is to derive the optimal boundary $B_u(f)$ for Y_u at time u given $F_u = f$. Indeed the optimal surrender policy at time u now depends on Y_u and F_u. We first work conditionally on F_u and assume that $F_u = f$ is given. We then look for the critical level for Y_u to trigger the optimal surrender of the policy. The surrender region is of the form $Y_u > B_u(f)$.[4]

To compute $e^g(Y_t, F_t, t)$ note that $f_{F_u}(f|Y_t, F_t) = f_{F_u}(f|F_t)$ is known in (3), and that the distribution of $f_{Y_u}(y|Y_t, F_t, F_u = f)$ is given in Proposition 4.2. Let us thus simplify the early exercise premium (21) as

$$\int_t^T e^{-r(u-t)} e^{-\kappa(T-u)} \int_0^\infty \int_{B_u(f)}^\infty y \left(r - \kappa + \frac{1}{u} \ln\left(\frac{y}{f}\right) \right) f_{Y_u}(y|Y_t, F_t, F_u = f) dy f_{F_u}(f|F_t) df du$$

and thus

$$e^g(Y_t, F_t, t) = e^{-\kappa T} e^{rt} \int_t^T e^{u(\kappa-r)} \int_0^\infty \left[\left(r - \kappa - \frac{\ln(f)}{u} \right) \mathbb{E}_1 + \frac{1}{u} \mathbb{E}_2 \right] f_{F_u}(f|F_t) df du$$

where $\mathbb{E}_1 := \mathbb{E}\left[\mathbb{1}_{Y > B_u(f)} Y \right]$ and $\mathbb{E}_2 := \mathbb{E}\left[\mathbb{1}_{Y > B_u(f)} Y \ln(Y) \right]$, and where Y is lognormal with log moments M_f and $\widehat{V}_{u,t}$ (mean and variance of $\ln(Y_u)|Y_t, F_t, F_u = f$ calculated in Proposition 4.2). It is then easy to prove that

$$\begin{cases} \mathbb{E}_1 = \Phi\left(\frac{M_f + \widehat{V}_{u,t} - \ln(B_u(f))}{\sqrt{\widehat{V}_{u,t}}} \right) e^{M_f + \frac{\widehat{V}_{u,t}}{2}} \\[2em] \mathbb{E}_2 = \sqrt{\widehat{V}_{u,t}} B_u(f)^{1+\frac{M_f}{\widehat{V}_{u,t}}} \frac{e^{-\frac{1}{2}\frac{M_f^2 + (\ln(B_u(f)))^2}{\widehat{V}_{u,t}}}}{\sqrt{2\pi}} + (M_f + \widehat{V}_{u,t}) \mathbb{E}_1 \end{cases}$$

This observation allows us to further simplify the early exercise premium to

$$e^g(Y_t, F_t, t) = e^{-\kappa T} e^{rt} \int_t^T e^{u(\kappa-r)} \int_0^\infty \left[\frac{\sqrt{\widehat{V}_{u,t}}}{u\sqrt{2\pi}} B_u(f)^{1+\frac{M_f}{\widehat{V}_{u,t}}} e^{-\frac{1}{2}\frac{M_f^2 + (\ln(B_u(f)))^2}{\widehat{V}_{u,t}}} \right.$$

$$\left. + \Phi\left(\frac{M_f + \widehat{V}_{u,t} - \ln(B_u(f))}{\sqrt{\widehat{V}_{u,t}}} \right) e^{M_f + \frac{\widehat{V}_{u,t}}{2}} \left(\frac{M_f + \widehat{V}_{u,t} - \ln(f)}{u} + (r - \kappa) \right) \right] f_{F_u}(f|F_t) df du$$

---

[4]See Appendix B.

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Replacing $B_u(f)$ by $\exp(\ln(B_u(f)))$, noting that $\hat{V}_{u,t}$ does not depend on f, and denoting by $H_u(x, f) := M_f + \hat{V}_{u,t} - \ln(x)$, this expression further simplifies to

$$e^g(Y_t, F_t, t) = e^{-\kappa T} e^{rt} \int_t^T e^{u(\kappa-r)} e^{\frac{\hat{V}_{u,t}}{2}} \int_0^\infty \left[\frac{\sqrt{\widehat{V}_{u,t}}}{u\sqrt{2\pi}} e^{-\frac{1}{2}\frac{H_u(B_u(f),f)^2}{\hat{V}_{u,t}}} \right.$$

$$\left. + \Phi\left(\frac{H_u(B_u(f),f)}{\sqrt{\widehat{V}_{u,t}}} \right) \left(\frac{H_u(f,f)}{u} + r - \kappa \right) \right] e^{M_f} f_{F_u}(f|F_t) df\, du$$

then

$$e^g(Y_t, F_t, t) = e^{-\kappa T} e^{rt} \int_t^T e^{u(\kappa-r)} e^{\frac{\hat{V}_{u,t}}{2}} Y_t^{\frac{t}{u}} F_t^{\frac{u-t}{2u}} \mathbb{E}\left[k(u, F_u, t) \right] du$$

where $k(u, f, t) = f^{\frac{u-t}{2u}} \left(\Phi\left(\frac{H_u(B_u(f),f)}{\sqrt{\widehat{V}_{u,t}}} \right) \left(\frac{H_u(f,f)}{u} + r - \kappa \right) + \frac{\sqrt{\widehat{V}_{u,t}}}{u\sqrt{2\pi}} e^{-\frac{1}{2}\frac{H_u(B_u(f),f)^2}{\hat{V}_{u,t}}} \right)$ and F_u is a LogNormal variable with density $f_{F_u}(f|F_t)$. $\qquad\square$

Theorem 2 provides a formula for the price of a VA with Asian benefits including a surrender option. However, since the early exercise premium depends on the optimal exercise boundary $B_t(f)$ it is not an explicit formula that can be implemented directly. One first needs to compute this boundary in analogy to Kim and Yu (1996). Note that the value of $B_T(F_T)$ at maturity is known and equal to $B_T(F_T) = G_T$. The procedure is then similar to the one-dimensional case except that one has a double integral to integrate.

To make the problem more tractable and reduce the number of equations to solve, we make the following assumption on the shape of the barrier. The benefit of this assumption appears clearly in Proposition 4.3 below, which describes the algorithm to derive the optimal surrender boundary.

Assumption 4.2. *Assume that the boundary $B_u(f)$ is given by the following form*

$$B_u(F_u) = \max(G_T e^{-r(T-u)}, a_u + b_u F_u) \tag{22}$$

At any time before maturity, it is never optimal to surrender unless the immediate payoff is at least equal to the discounted value of the minimum terminal payoff G_T as this is the minimum amount guaranteed at time T. We also know that along the exercise boundary it holds

$$V(F_t, t) = e^{-\kappa(T-t)} Y_t = B_t(F_t) = \max(G_T e^{-r(T-t)}, a_t + b_t f).$$

Thus, by formula (20)

$$B_t(F_t) = v^g(\max(G_T e^{-r(T-t)}, a_t + b_t F_t), F_t, t) + e^g(\max(G_T e^{-r(T-t)}, a_t + b_t F_t), F_t, t).$$

This is an integral equation for the optimal exercise boundary because of the form of $e^g(\cdot, \cdot, \cdot)$ in (21). Observe, however, that in order to compute $\max(G_T e^{-r(T-t)}, a_t + b_t F_t)$ at time t, the optimal exercise boundary for future times must be known. Since it holds that $B_T = G_T$ ($b_T = 0$) at expiration, we can work backwards through time to recursively recover the optimal exercise boundary. Because formula (21) does not have an analytic solution, numerical integration schemes must be used. Practically this is done by dividing the interval $[0, T]$ into n equidistant subintervals $0 = t_0 < t_1 < ... < t_n = T$, where the times t_i, $i = 0, .., n$, represent the possible early exercise times. Define $g(u) := e^{-\kappa T} e^{rt} e^{u(\kappa - r)} e^{\frac{\widehat{V}_{u,t}}{2}} Y_t^{\frac{t}{u}} F_t^{\frac{u-t}{2u}} \mathbb{E}[k(u, F_u, t)]$. Then, the integral in (21) is approximated by

$$I(k) = \frac{T}{n} \sum_{i=1}^{k-1} g(t_{n-i}), \quad k = 1, .., n. \tag{23}$$

Note, that at time t_{n-1}, $I(1) = 0$.

Proposition 4.3 (Derivation of the optimal exercise boundary). *The following backward procedure allows to derive the exercise boundary approximately.*

- $B_{t_n} = B_T = G_T$, $b_T = 0$.

- *Recursively, for $k = 1..n$:*

    - *For m values of F_t, compute the optimal boundary $B_t(F_t)$ using (23) and solving*

    $$B_t(F_t) = v^g(\max(G_T e^{-r(T-t)}, a_t + b_t F_t), F_{t_{n-k}}, t_{n-k}) + I(k).$$

    - *Out of the m values obtained, use those above $G_T e^{-r(T-t)}$ to perform a linear regression and obtain a_t and b_t.*

A numerical illustration is given in the next section. Note that the technique described in this section will apply for other types of path-dependent benefits. The derivation holds when at any time u, conditional on the value of the underlying fund $F_u = f$ at time u, the optimal strategy is driven by checking whether some other quantity (here the geometric average) is above a level $B_u(f)$ (in other words the optimal strategy is a threshold strategy conditionally on the fund value at time u). Finally. note that the approximation (22) significantly simplifies the implementation as it locally approximates the surrender boundary with a piecewise linear function. From our numerical experiments we found that this is a satisfactory approximation.

5 Numerical Results

This section presents some numerical examples to illustrate the techniques presented in Sections 3 and 4 respectively.

5.1 Optimal Boundary for the VA studied in Section 3

We perform a sensitivity analysis to further shed light on some properties of the exercise boundary derived in Section 3. Unless stated otherwise, we assume that $\kappa = 0$, $r = 0.03$, $\sigma = 0.2$ and $T = 15$ (years). The guaranteed amount G_T is equal to 100, and $g = 0$ so that $G_t = G_T$ at any time t. We find that the fair fee $c^* = 0.91\%$ neglecting the surrender benefit.

Figure 1 shows optimal exercise boundaries for the set of parameters given above when varying one parameter at a time. There are a few things to be noticed. First, as discussed earlier, the time zero value of the boundary is greater than the fund value at time 0 and the value at maturity T is equal to the guarantee G_T. Secondly, the graph of the exercise boundary is generally non-monotonic. The curve slowly increases to its maximum and then declines rapidly to G_T.

In the following we examine the sensitivity of the optimal exercise boundary with respect to the parameters σ, r, c, T, G_T and κ. Panel A of Figure 1 illustrates the sensitivity with respect to the volatility σ. We compute the exercise boundary for values of $\sigma = 15\%, 20\%, 25\%$ and 30%. We observe that as volatility increases the optimal exercise boundary gets pushed further up. With a high volatility, the policyholder would surrender the contract at higher values of the underlying fund than if he had invested in a fund with a lower volatility. Intuitively this result can be explained by the fact that the fund fluctuates more heavily if the volatility is higher. Therefore, the maturity benefit is more valuable.

Panel B of Figure 1 displays the sensitivity with respect to the risk-free interest rate r. We vary the interest rate between 2% and 3.5% and compute the optimal exercise boundary. Similar to the sensitivity with respect to the volatility, we observe that the optimal exercise boundary is higher for higher interest rates. However, the extent of the difference is smaller, and the boundary is not as sensitive towards the interest rate as to the volatility.

In Panel C of Figure 1 we show the sensitivity of the optimal exercise boundary of the sensitivity analysis with respect to the fee c. Since insurance companies don't always charge the fair fee, it is interesting to investigate what happens if the fee is somewhat higher or lower. In our case, the fee takes values from 0.5% to 2.0%. The figure shows that with a higher fee the optimal exercise boundary is lower. This is intuitive since with a higher fee, the policyholder has to pay more for the guarantee. Thus, the mismatch between the premium for the guarantee and its value is even greater resulting in earlier exercise times. This also increases the value of the surrender option, showing that increasing c is not a good way to pay for surrender benefits (see also Milevsky and Salisbury (2001)). We also observe that the optimal exercise boundary is very sensitive to changes in the fee. From an initial exercise value of 150 at time zero for the fair fee, the exercise value drops to about 115 for a fee of 2.0%. Likewise if the fee is reduced to 0.5% the exercise value increases to just above 180.

Panel D of Figure 1 shows the sensitivity with respect to the maturity T. It illustrates that with increasing maturity the optimal exercise boundary increases as well. Considering a short time to maturity the fund value is less likely to reach high values. It is also known that the price of plain vanilla options are negatively correlated with the time to maturity, i.e. it loses value the closer it gets to maturity. Therefore, if we decrease the maturity T the option is worth less and should thus be surrendered at a lower fund level.

We analyze the sensitivity of the exercise boundary with respect to the guarantee G_T in Panel E of Figure 1. For $G_T = 75, 100, 125$ and 150 we compute the optimal exercise boundary B_t. The graphs look quite different from the ones above. We observe that the higher the guarantee the lower is the initial value of the exercise boundary. However at the same time the slope is higher for graphs with a higher guarantee. This effect can be explained by considering the fees c^* displayed in the table below. The fee for a contract with a guarantee of 150 is about 15 times greater than the fee of a contract with a guarantee of 75. So on the one hand the policyholder has a high guaranteed return at maturity. But on the other hand he has to pay a high fee for it. For this reason, it is better for the policyholder to surrender the contract earlier than if he had a lower guarantee implying a lower fee.

| G_T	75	100	125	150
c^*	0.35%	0.91%	2.02%	5.28%

Lastly, Panel F of Figure 1 represents the effect of κ. The optimal boundary quickly moves up when κ increases: the surrender incentive is much lower because of the surrender penalty. In practice κ can be chosen high enough to have very low surrender incentives. Throughout our study we assumed $\kappa = 0$ to find the maximum risk for companies if they do not charge for the surrender option.
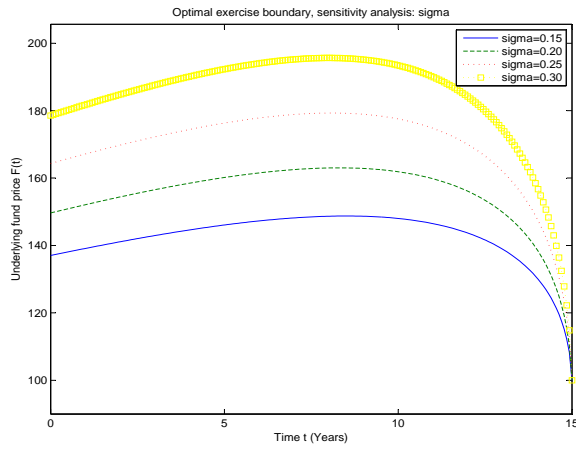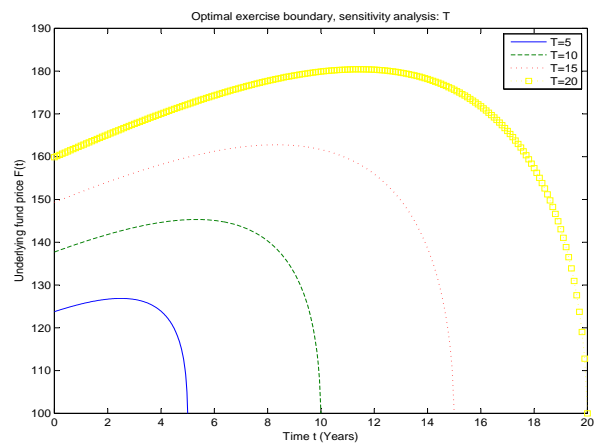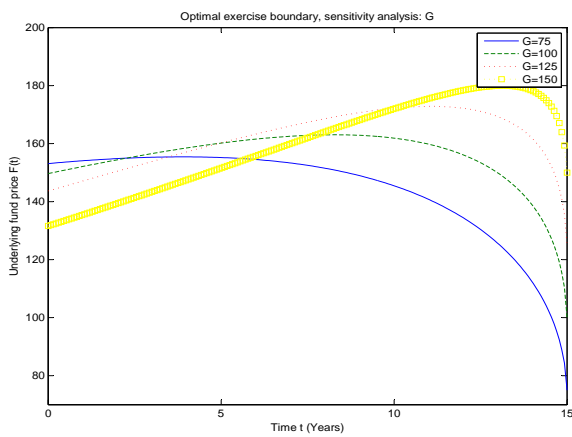
Panel A: Sensitivity to σ
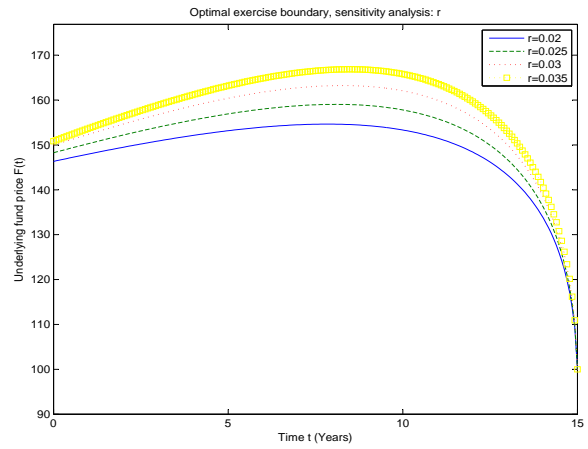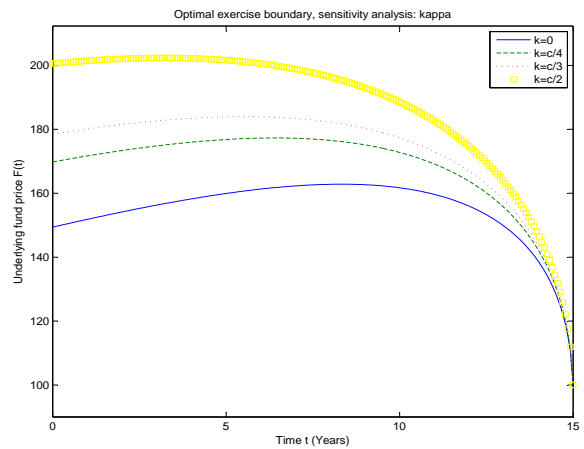
Panel B: Sensitivity to r
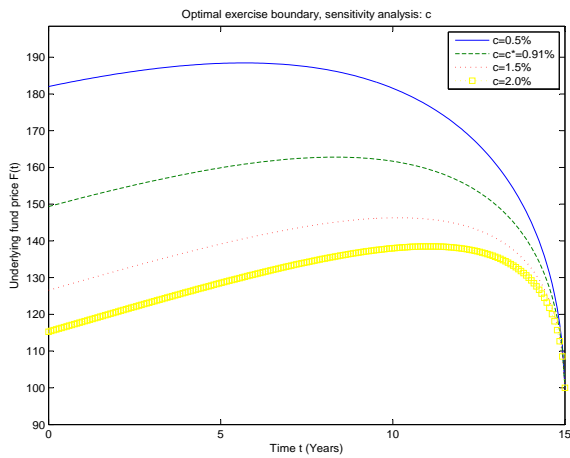
Panel C: Sensitivity to c

Panel D: Sensitivity to T

Panel E: Sensitivity to G_T

Panel F: Sensitivity to κ

Figure 1: Sensitivity Analysis: The fee rates are computed to make the contract with the European benefit fair in all panels except in Panel C in which the sensitivity to the fee rate c is studied.

5.2 Optimal Boundary for the VA studied in Section 4

We illustrate the shape of the optimal boundary for a VA with path-dependent payoff in Figure 2. Since the optimal boundary at time t, B_t, depends on time t and on the value of the fund F_t, the optimal surrender boundary throughout the life of the contract must thus be represented by a surface (Figure 2). It is also possible to fix a value F_t and obtain a curve which shows the evolution of the boundary through time (as it is in Figure 3). We consider a 10-year contract with payoff $\max(Y_T, G_T)$ as defined in Section 4. Here, we assume that the guaranteed roll-up rate is 0.025 so that $G_T = F_0 e^{0.025T}$. We also assume that there is no surrender charges. Neglecting the surrender benefit, we use the fair fee $c^* = 0.0197$. Market assumptions are as in the previous section.
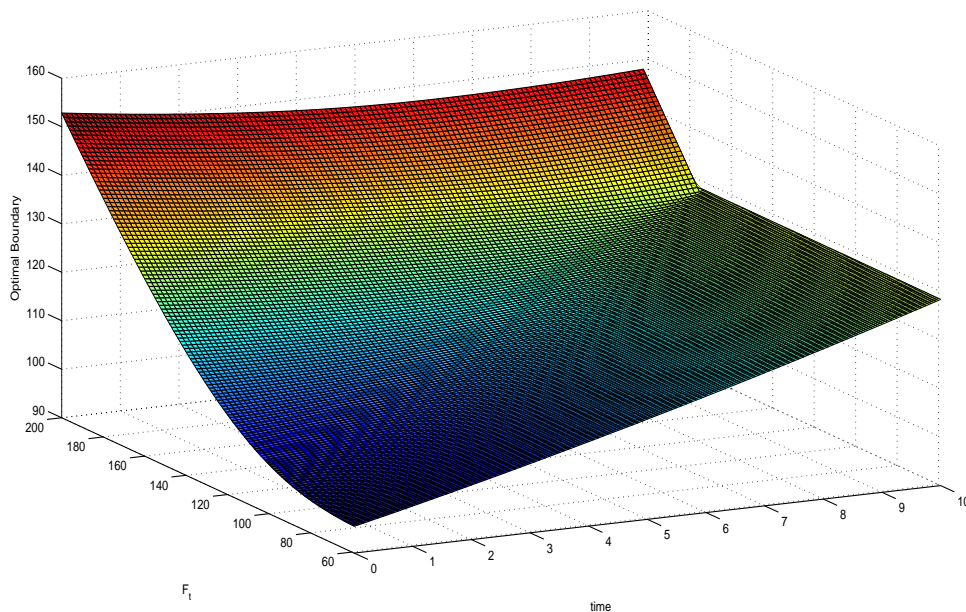

Figure 2: Optimal exercise surface for a 10-year geometric average VA with $G_T = F_0 e^{0.025 \times 10}$ and $\kappa = 0$.

For high values of F_T, the boundary drops at maturity, because for any value $Y_T > G_T$, the option is exercised. However, before maturity, it is not necessarily optimal to surrender because the average of the fund might still increase. This is especially the case when $F_t > Y_t$. For low values of F_t, the boundary is close to $G_T e^{-r(T-t)}$, the discounted value of G_T. When the fund value is low, it drags the average down and decreases the probability that the average

at maturity is above the guarantee. Thus, for low values of F_t, it may be optimal to lapse the contract and "cash in the gains" earlier. In general, for a fixed time t, this causes the boundary to increase with F_t. This behaviour is more noticeable at the beginning of the contract since there is more time for the average to increase. The optimal surrender boundaries are relatively low, because the average is a lot less volatile than the fund. For this reason, it is often optimal to surrender early, even when the fund value is high, because the expected increase in the average is less than the risk-free rate. Thus, it would be optimal to withdraw the amount of the average and invest it at the risk-free rate. In fact, when Y_t increases past the $G_T e^{-r(T-t)}$, the value of the option drops quickly because of the low volatility of the average. For this reason, the optimal surrender boundaries are quite low, even for high values of the fund. This indicates that average-type maturity benefits tend to increase the value of the surrender option.
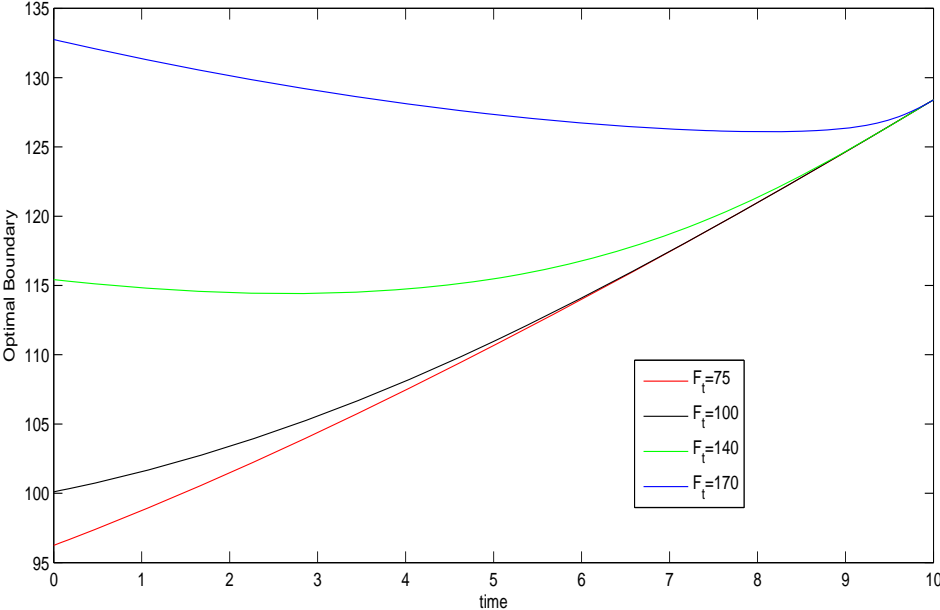

Figure 3: Optimal exercise boundary for a 10-year geometric average VA as a function of time for different values of F_t with $G_T = F_0 e^{0.025 \times 10}$ and $\kappa = 0$.

6   Concluding Remarks

In this paper, we presented a method that allows us to derive a formula for the price of a VA contract. We do so by decomposing it into a corresponding European part and an early exercise premium. However, in order to use this pricing formula the optimal exercise boundary needs to be known. Subsequently, we found that it fulfills an integral equation that can be solved recursively going backwards in time. We implemented these formulae and performed some numerical examples. They revealed that the optimal exercise boundary is a non-monotonic function which increases at first and then decreases to finally attain the guaranteed amount at maturity. By performing sensitivity analysis we found that with increasing volatility, interest rate, surrender charge and maturity the optimal exercise boundary is pushed up. If we increase the guarantee, however, we find a lower boundary at the beginning. But due to a higher slope the boundary takes higher values as maturity is approached before dropping back to the guaranteed level. This effect is explained by higher fair fees for contracts with a high guarantee.

Our method is general enough to be used when the benefits are path-dependent. We considered the geometric average of the fund as an example of such a payoff. Analogously, we derived a pricing formula and an integral equation for the optimal exercise boundary which depends on the geometric average as well as on the fund value itself. We found that in general if the fund value is larger than the geometric average it might not be optimal to surrender, but to wait as the fund will increase the geometric average. Surrender incentives tend thus to be reduced by the presence of Asian benefits.

In this paper, we assumed that the underlying follows a geometric Brownian motion. Although this model is too simple to fit actual market data, it is sufficient to shed some light on the different factors influencing the optimal surrender boundary. Since the transition density of the underlying asset is known explicitly, we are able to obtain integral representations for the value of the surrender option. Our method can easily be extended to other market models as long as the model guarantees the existence of a portfolio that replicates the fund value using traded assets. In the case when the transition density is not known in explicit form, the method can still be used, without deriving an analytical form for the integrand but approximating it by Monte Carlo techniques for instance. Thus, our method can be extended to obtain the surrender boundary under more realistic market models.

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A   Optimal Surrender Region for GMAB

We prove here that the optimal surrender strategy for a GMAB is a threshold strategy. That is, we show that when the surrender charge is of the form $\kappa_t = 1 - e^{-\kappa(T-t)}$, $\kappa < c$, then for any time t before maturity, there exists a value F_t^* above which the value of the contract is less than the surrender benefit available immediately. This proof is inspired by Section 3 of Wu and Fu (2003). We let τ be a stopping time with respect to $\mathcal{F}_t$ and denote by $\mathcal{T}_t$ the set of all stopping times τ greater than t and bounded by T. We express the value at time t of the variable annuity contract $V(x,t)$ by

$$V(x,t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)} \psi(F_\tau, \tau) | F_t = x \right],$$

where $x = F_t$ and

$$\psi(x,t) = \begin{cases} e^{-\kappa(T-t)}x, & \text{if } 0 \leqslant t < T \\ \max(x, G), & \text{if } t = T. \end{cases} \tag{24}$$

We also define the optimal surrender region at time t, denoted $R^*(t)$, by

$$R^*(t) = \left\{ F_t : \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)} \psi(F_\tau, \tau) | \mathcal{F}_t \right] \leqslant \psi(F_t, t) \right\}. \tag{25}$$

In order to obtain the form of the optimal exercise region defined in (25), we first rewrite $R^*(t)$ as

$$R^*(t) = \left\{ F_t : \frac{V(F_t, t)}{\psi(F_t, t)} \leqslant 1 \right\}.$$

We analyze the function $\gamma(x,t) := \frac{V(x,t)}{\psi(x,t)}$ in Lemma A.1.

Lemma A.1. *Let* $\gamma(x,t) = \frac{V(x,t)}{\psi(x,t)}$ *for* $t \in [0,T]$. *Then,*

- *For* $t = T$, $\gamma(x,T) = 1$.

- *For* $t \in [\, 0, T\,)$, $\gamma(x,t)$ *is non-increasing in* x.

Proof. At $t = T$, we have that $\frac{V(F_T,T)}{\psi(F_T,T)} = \frac{\psi(F_T,T)}{\psi(F_T,T)} = 1$. For $t \in [\, 0, T\,)$, note that $\gamma(x,t)$ can be

rewritten as

$$\gamma(x,t) = \frac{V(x,t)}{\psi(x,t)} = \frac{\sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\psi(F_\tau,\tau)\big|\, F_t = x\right]}{e^{-\kappa(T-t)}x}$$

$$= \frac{\sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}e^{-\kappa(T-\tau)}F_\tau + e^{-r(\tau-t)}(G-F_\tau)^+ \mathbb{1}_{\{\tau=T\}}\big|\, F_t = x\right]}{xe^{-\kappa(T-t)}}$$

$$= \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-(r-\kappa)(\tau-t)+X_{\tau-t}} + e^{-r(\tau-t)}\left(\frac{G}{xe^{-\kappa(T-t)}} - e^{\kappa(\tau-t)+X_{\tau-t}}\right)^+ \mathbb{1}_{\{\tau=T\}}\bigg|\, F_t = x\right],$$

where $X_{\tau-t} = (r - c - \frac{\sigma^2}{2})(\tau - t) + \sigma(W_\tau - W_t)$. Observe that x is a positive real number and that $\mathbb{E}\left[e^{-r(\tau-t)}\psi(F_\tau,\tau)|F_t = x\right]$ is finite, which allows us to take $\frac{1}{x}$ inside the supremum. Now fix a time $u \in (t, T]$ and observe that

$$\mathbb{E}\left[e^{-(r-\kappa)(u-t)+X_{u-t}} + e^{r(u-t)}\left(\frac{G}{xe^{-\kappa(T-t)}} - e^{\kappa(\tau-t)+X_{u-t}}\right)^+ \mathbb{1}_{\{u=T\}}\bigg|\, F_t = x+\varepsilon\right]$$

$$= \mathbb{E}\left[e^{-(r-\kappa)(u-t)+X_{u-t}} + e^{r(u-t)}\left(\frac{G}{(x+\varepsilon)e^{-\kappa(T-t)}} - e^{\kappa(\tau-t)+X_{u-t}}\right)^+ \mathbb{1}_{\{u=T\}}\right]$$

$$\leqslant \mathbb{E}\left[e^{-(r-\kappa)(u-t)+X_{u-t}} + e^{r(u-t)}\left(\frac{G}{xe^{-\kappa(T-t)}} - e^{\kappa(\tau-t)+X_{u-t}}\right)^+ \mathbb{1}_{\{u=T\}}\right]$$

$$= \mathbb{E}\left[e^{-(r-\kappa)(u-t)+X_{u-t}} + e^{r(u-t)}\left(\frac{G}{F_t e^{-\kappa(T-t)}} - e^{\kappa(\tau-t)+X_{u-t}}\right)^+ \mathbb{1}_{\{u=T\}}\bigg|\, F_t = x\right],$$

where the third line results from the fact that for $a > b \geqslant 0$ and $c > 0$, $(b-c)^+ \leqslant (a-c)^+$. Then, since any $\tau \in \mathcal{T}_t$ takes values in $(t, T]$ with probability 1, the inequality holds almost surely for any $\tau \in \mathcal{T}_t$. In other words, we have that

$$\frac{\mathbb{E}\left[e^{r(\tau-t)}\psi(F_\tau,\tau)|F_t = x+\varepsilon\right]}{x+\varepsilon} \leqslant \frac{\mathbb{E}\left[e^{r(\tau-t)}\psi(F_\tau,\tau)|F_t = x\right]}{x}, \qquad \text{a.s.}$$

for any $\tau \in \mathcal{T}_t$. Taking the supremum over all stopping times τ on both sides, we obtain $\gamma(x+\varepsilon,t) \leqslant \gamma(x,t)$ for all $t \in [\,0,T\,)$. $\qquad\square$

This result allows us to say that if we can find F_t^* such that $\gamma(F_t^*, t) = 1$, then for any $F_t \geqslant F_t^*$, $\gamma(F_t, t) \leqslant 1$ and for any $F_t < F_t^*$, $\gamma(F_t, t) > 1$. Thus, the optimal surrender region $R^*(t)$ has the form $[\,F_t^*, \infty\,)$.

Using Lemma A.1, we obtain the following theorem, which confirms that under certain assumptions, we can always find such F_t^*, so that the optimal surrender strategy is of the threshold type.

Theorem A.1. *The optimal exercise strategy for the surrender option is to surrender the contract when $F_t \geqslant B_t$, with*

$$B_t = \inf\{x : V(x,t) \leqslant \psi(x,t)\},$$

for $t \in [\,0,T\,)$. When the surrender charges are of the form $\kappa_t = 1 - \exp(-\kappa(T-t))$ with $\kappa < c$, then $B_t < \infty$ for all $t \in [0,T]$.

Proof. We show that for any $t \in [\,0,T\,)$, it is possible to find x such that $V(x,t) \leqslant \psi(x,t)$. Note that for $t \in [\,0,T\,)$, $\psi(x,t) = xe^{-\kappa(T-t)}$. Thus, we need to show that it is possible to find x^* such that

$$V(x^*,t) \leqslant x^* e^{-\kappa(T-t)}.$$

Then, by Lemma A.1, the inequality will hold for any $x > x^*$. First, fix $t \in [\,0,T\,)$ and observe that for any stopping time $\tau \in \mathcal{T}_t$, we have

$$\mathbb{E}[e^{-r(\tau-t)}\psi(F_\tau,\tau)|F_t = x] = \mathbb{E}[e^{-r(\tau-t)}F_\tau e^{-\kappa(T-\tau)}|F_t = x] + \mathbb{E}[e^{-r(T-t)}(G-F_T)^+\mathbb{1}_{\{\tau=T\}}|F_t = x]$$

$$= \mathbb{E}[xe^{-\kappa(T-\tau)}e^{-(c+\frac{\sigma^2}{2})(\tau-t)+\sigma(W_\tau-W_t)}|F_t = x] + \mathbb{E}[e^{-r(T-t)}(G-F_T)^+\mathbb{1}_{\{\tau=T\}}|F_t = x]$$

$$\leqslant \mathbb{E}[\mathbb{E}[xe^{-\kappa(T-\tau)}e^{-(c+\frac{\sigma^2}{2})(s-t)+\sigma(W_s-W_t)}|\tau = s]|F_t = x] + \mathbb{E}[e^{-r(T-t)}(G-F_T)^+|F_t = x]$$

$$\leqslant \mathbb{E}[xe^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x] + e^{-c(T-t)}P(x,t),$$

where $P(x,t)$ is the of a European put option on the fund with $F_t = x$, with maturity T and strike G. To obtain the last line, we use $\mathbb{E}\left[e^{-\frac{\sigma^2}{2}s+\sigma W_s}\right] = 1$. Now, we need to show that for any $\tau \in \mathcal{T}_t$, we can find x such that

$$\mathbb{E}[xe^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x] + e^{-c(T-t)}P(x,t) < xe^{-\kappa(T-t)}.$$

We know that

$$\lim_{x\to\infty} P(x,t) = 0.$$

Then, for any $\varepsilon > 0$, there exists x^* (large enough) so that $P(x^*,t) < \varepsilon e^{c(T-t)}$. Thus, for x^*,

$$\mathbb{E}[x^* e^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x] + e^{-c(T-t)}P(x^*,t) < \mathbb{E}[x^* e^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x] + \varepsilon.$$

Since this holds for any $\varepsilon > 0$, we can get arbitrarily close to $\mathbb{E}[x^* e^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x]$. Since $\kappa < c$,

$$\mathbb{E}[x^* e^{-\kappa(T-\tau)}e^{-c(\tau-t)}|F_t = x] < \mathbb{E}[x^* e^{-\kappa(T-t)}|F_t = x] = x^* e^{-\kappa(T-t)}.$$

Thus, for any $x > x^*$, we have

$$\mathbb{E}[e^{-r(\tau-t)}\psi(F_\tau,\tau)|F_t = x] < xe^{-\kappa(T-t)}.$$

Taking the supremum over all stopping times on both sides, we get $V(x,t) \leqslant \psi(x,t)$, which ends the proof. $\qquad\square$

B Optimal Surrender Region with Asian Benefits

We prove here that the optimal surrender strategy for the path-dependent payoff introduced in Section 4 is also a threshold strategy. That is, we show that when the surrender charge is of the form $\kappa_t = 1 - e^{-\kappa(T-t)}$, $\kappa < c$, and satisfies the conditions stated at the beginning of Section 4, then for any time t before maturity and any value F_t, there exists a geometric average Y_t^* above which the value of the contract is less than the surrender benefit available immediately. This proof is similar to the one presented in Appendix A. We let τ and $\psi(x, t)$ be defined as in Appendix A. We express the value of the variable annuity contract $V(Y_t, F_t, t)$ by

$$V(Y_t, F_t, t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\psi(Y_\tau, \tau)|Y_t, F_t\right].$$

We also define the optimal surrender region at time t, denoted $R^*(F_t, t)$, by

$$R^*(F_t, t) = \left\{Y_t : \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\psi(Y_\tau, \tau)|\mathcal{F}_t\right] \leqslant \psi(Y_t, t)\right\}. \qquad (26)$$

We can also rewrite $R^*(F_t, t)$ as

$$R^*(F_t, t) = \left\{Y_t : \frac{V(Y_t, F_t, t)}{\psi(Y_t, t)} \leqslant 1\right\}.$$

We analyze the function $\gamma^g(x, F_t, t) \equiv \frac{V(x, F_t, t)}{\psi(x, t)}$ and obtain Lemma B.1.

Lemma B.1. *Let $\gamma^g(x, F_t, t) = \frac{V(x, F_t, t)}{\psi(x, t)}$ for $t \in [0, T]$. Then,*

- *For $t = T$, $\gamma^g(x, F_T, T) = 1$.*

- *For $t \in [\, 0, T\,)$, $\gamma(x, F_t, t)$ is non-increasing in x.*

Proof. To prove this lemma, we use the fact that $Y_u|F_t, Y_t$ has the same distribution as $Y_t^{\frac{t}{u}} F_t^{\frac{u-t}{u}} e^{\mu(t,u)+\sigma(t,u)Z}$, where Z is a standard normal random variable, $\mu(t, u) = \frac{r-c-\frac{\sigma^2}{2}}{2u}(u-t)^2$ and $\sigma^2(t, u) = \frac{\sigma^2}{3u^2}(u-t)^3$. The rest of the proof is similar to the proof of Lemma A.1. $\qquad \square$

In order to prove that the optimal surrender strategy is of the threshold type, we need to show that for any t, F_t, $0 \leqslant t < T$, $F_t > 0$, there exists a value Y_t^* such that $V(Y_t, F_t, t) \leqslant \psi(Y_t, t)$. It is shown in Theorem B.1.

Theorem B.1. *The optimal exercise strategy for the path-dependent surrender option is to surrender the contract when $Y_t \geqslant B_t(f)$, with*

$$B_t(f) = \inf\{x : V(x, f, t) \leqslant \psi(t, x)\},$$

for $t \in [\, 0, T\,)$, $f > 0$. $B_t(f) < \infty$ for all $t \in [0, T]$, $f > 0$ if the surrender charges are of the form $\kappa_t = 1 - \exp(-\kappa(T - t))$ and satisfy $\kappa < \frac{r + c + \frac{\sigma^2}{6}}{2}$, and $c < r - \frac{\sigma^2}{6}$.

Proof. We show that for any $t \in [\, 0, T\,)$, $f > 0$ it is possible to find x such that $V(x, f, t) \leqslant \psi(t, x)$. Note that for $t \in [\, 0, T\,)$, $\psi(t, x) = xe^{-\kappa(T-t)}$. Thus, we need to show that it is possible to find x such that
$$V(x, f, t) \leqslant xe^{-\kappa(T-t)}.$$

First, fix $t \in [\, 0, T\,)$ and observe that for any stopping time $\tau \in \mathcal{T}_t$, we have

$\mathbb{E}[e^{-r(\tau - t)}\psi(F_\tau, \tau)|F_t = x]$
$$= \mathbb{E}[e^{-r(\tau - t)}Y_\tau e^{-\kappa(T-\tau)}|Y_t = x, F_t = f] + \mathbb{E}[e^{-r(T-t)}(G - Y_T)^+ \mathbb{1}_{\{\tau = T\}}|Y_t = x, F_t = f]$$
$$\leqslant \mathbb{E}[e^{-r(\tau - t)}Y_\tau e^{-\kappa(T-\tau)}|Y_t = x, F_t = f] + \mathbb{E}[e^{-r(T-t)}(G - Y_T)^+|Y_t = x, F_t = f]$$

The second term of the equation is simply the price of a geometric Asian put option with strike G. This term goes to 0 as $x \to \infty$ (see for example Kemna and Vorst (1990)). Now by the same reasoning as in the proof of Theorem A.1, it suffices to show that there exists x^* such that
$$\mathbb{E}[e^{-r(\tau - t)}Y_\tau e^{-\kappa(T-\tau)}|Y_t = x^*, F_t = f] < x^* e^{-\kappa(T-t)}.$$

Then, by Lemma B.1, the inequality will hold for any $x > x^*$. For a fixed $t \in [0, T)$, $f \in (0, \infty)$, this can be done by taking any $x > f$. Let $f < x^* < \infty$. Then, by first conditioning on the stopping time τ, we have

$\mathbb{E}[e^{-r(\tau - t)}Y_\tau e^{-\kappa(T-\tau)}|Y_t = x^*, F_t = f]$
$$= \mathbb{E}\big[e^{-\kappa(T-\tau)}e^{-r(\tau - t) + \frac{t}{\tau}\ln x^* + \frac{\tau - t}{\tau}\ln f + \frac{r - c - 0.5\sigma^2}{2\tau}(\tau - t)^2 + \frac{\sigma^2(\tau - t)^3}{6\tau^2}}\big]$$
$$< \mathbb{E}\big[x^* e^{-\kappa(T-\tau)}e^{-r(\tau - t) + \frac{r - c - 0.5\sigma^2}{2\tau}(\tau - t)^2 + \frac{\sigma^2(\tau - t)^3}{6\tau^2}}\big]$$
$$< \mathbb{E}\big[x^* e^{-\kappa(T-\tau)}e^{-r(\tau - t) + \frac{r - c - \sigma^2/6}{2\tau}(\tau - t)^2}\big]$$
$$< \mathbb{E}\big[x^* e^{-\kappa(T-\tau)}e^{-\frac{1}{2}(r + c + \frac{\sigma^2}{6})(\tau - t)}\big]$$
$$< \mathbb{E}\big[x^* e^{-\kappa(T-\tau)}e^{\kappa(\tau - t)}\big]$$
$$= x^* e^{-\kappa(T-t)}$$

To get the fourth and the fifth line, we use the assumption $c < r - \frac{\sigma^2}{6}$ and the fact that $\tau > \tau - t$. By taking the supremum over all stopping times, this allows us to conclude that under our assumptions, it is always possible to find an average fund value x such that
$$V(x, f, t) \leqslant xe^{-\kappa(T-t)}.$$

$\square$